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Natural decomposition approximation solution for second order nonlinear differential equations

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Abstract
In this article, I introduce the Natural Decomposition Method (NDM) to solve nonlinear second order differential equations. I use the NDM to obtain exact solutions for three different types of nonlinear second order ordinary differential equations. The NDM is the combination of Natural transform method (NTM) and the Adomian decomposition method (ADM). The proposed method gives exact solutions in the form of a rapid convergence series and applied directly without using any linearization, discretization, transformation, or taking some restrictive assumptions. Therefore, the Natural Decomposition Method is an outstanding mathematical tool for solving nonlinear second order ordinary differential equation. One can conclude that the NDM is efficient and easy to use.

Keywords: Natural Decomposition Method, approximation solution, Adomian decomposition method

Introduction

In the last several years with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. Nonlinear evolution equations are widely used to describe many important phenomena and dynamic process in physics, mechanics, chemistry, biology and so on. Most of the differential equations arise in a large number of mathematical and engineering applications. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear differential equations. Different researchers introduced integral transforms and explored their application in solving both ordinary and partial differential equation problems arising from different disciplines.

In this article, I suggested a technique called the Natural Decomposition Method (NDM) for solving the nonlinear second order ordinary differential equations [1]. There are many integral transform methods [2, 3-9] exists in the literature to solve ODEs. The most used one is the Laplace transformation [10]. Other methods used recently to solve PDEs and ODEs, such as, the Sumudu transform [11], the Reduced Differential Transform Method (RDTM) [12-15] and the Elzaki transform [4-9]. Fethi Belgacem and R. Silambarasan [16, 17], used the N–Transform to solve the Maxwell's equation, Bessel's differential equation and linear and nonlinear Klein Gordon Equations and more. Also, Zafar H. Khan and Waqar A. Khan [18], used the N–Transform to solve linear differential equations.

The new technique is combination of the Natural transform Method (NTM) and Adomian Decomposition Method (ADM). The Adomian decomposition method (ADM) [19, 20], proposed by George Adomian, has been applied to a wide class of linear and nonlinear PDEs. The proposed technique lead to approximate or analytical solution in form of a rapidly convergence series, and is applied directly without any unnecessary linearization, discretization, transformation or taking some restrictive assumptions. The new technique is used successfully to obtained exact solution of the equation governing nonlinear second order ordinary differential equations. This shows the reliability and simplicity of the new technique. Thus, the Natural Decomposition Method is a powerful mathematical technique for solving a wide class of nonlinear ordinary differential equations.

Definitions and Properties of the N-Transform

The natural transform of the function $f(t)$ for $\in (-\infty, \infty)$, then the general integral transform is defined by [18, 21]:

$\mathcal{J}[f(t)](s) = \int_{-\infty}^{\infty} k(s,t) f(t)\,dt$, where $k(s, t)$ represent the kernel of the transform, s is the real or complex number which is independent of t. In this paper we work this transform to solve initial value problems f constant coefficients.
The natural transform of the function $f(t)$ for $t \in (-\infty, \infty)$ is defined by [1],

$$\mathcal{N}[f(t)] = M(s,u) = \int_{-\infty}^{\infty} f(ut) e^{-st}\,dt \ ; s,u \in (-\infty,\infty) \tag{2.1}$$

$$= \int_{-\infty}^{0} f(ut) e^{-st}\,dt + \int_{0}^{\infty} f(ut) e^{-st}\,dt$$

$$= \mathcal{N}^{-}[f(t)] + \mathcal{N}^{+}[f(t)]$$

Then we define the Natural transform (N-transform)

$$\mathcal{N}[f(t)] = \mathcal{N}^{+}[f(t)] = \int_{0}^{\infty} f(ut) e^{-st}\,dt \ , s,u \in (-\infty,\infty) \tag{2.2}$$

The original function $f(t)$ in equation (1) is called the inverse transform which is denoted by

$$f(t) = \mathcal{N}^{-1}\{M(s,u)\}$$

Note: Note if $u = 1$, then Eq. (2) can be reduced to the Laplace transform and if $s = 1$, then Eq. (2) can be reduced to the Sumudu transform. Now we give some of the N–Transforms and the conversion to Sumudu and Laplace [16, 17].

Table 1: Special N-Transforms and the Conversion to Sumudu [23-25] and Laplace Transform

$f(t)$	$\mathcal{N}[f(t)]$	$S[f(t)]$	$L[f(t)]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
sint	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$
cost	$\frac{s}{s^2+u^2}$	$\frac{1}{1+u^2}$	$\frac{s}{1+s^2}$
$t^n e^{at}$	$\frac{n!\,u^n}{(s-au)^{n+1}}$	$\frac{n!\,u^n}{(1-au)^{n+1}}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{t^{n-1}}{(n-1)!}, n = 1,2,\ldots$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$

Table 2: Properties of N-Transforms

Functional form	**Natural transform**
$f(t)$	$F(s,u)$
$N[af(t) \pm bg(t)]$	$aF(s,u) \pm bG(s,u)$
$N[af(t)]$	$\frac{1}{a}F(s,u)$
$N[f'(t)]$	$\frac{s}{u}F(s,u) - \frac{f(0)}{u}$
$N[f''(t)]$	$\frac{s^2}{u^2}F(s,u) - s\frac{f(0)}{u^2} - \frac{f'(0)}{u}$
$N\left[\int_0^t f(p)\,dp\right]$	$\frac{u}{s}F(s,u)$

Methodology of the Natural Decomposition Method

In this section, i illustrate the applicability of the Natural Decomposition Method to some nonlinear second ordinary differential equations.

Consider the general nonlinear ordinary differential equation of the form:

$$L(y) + R(y) + F(y) = g(t) \tag{3.1}$$

Subject to the initial condition

$$y(0) = h(t) \tag{3.2}$$

Where L is an operator of the highest derivative, R is the reminder of the differential operator, $g(t)$ is nonhomogeneous term and $F(y)$ is the nonlinear term.

Suppose L is a differential operator of the first order, then by taking the natural transform i.e., N- Transform of eq. (3), we have,

$$\frac{sY(s,u)}{u} - \frac{y(0)}{u} + N^{+}[R(y)] + N^{+}[F(y)] = N^{+}[g(t)] \tag{3.3}$$

By substituting Eq.(4) into Eq. (5),we obtain

$$Y(s,u) = \frac{h(t)}{s} + \frac{u}{s}N^{+}[g(t)] - \frac{u}{s}N^{+}[R(y) + F(y)] \tag{3.4}$$

Taking the inverse of the N-Transform of Eq.(6), we have

$$y(t) = G(t) - N^{-1}\left[\frac{u}{s}N^{+}[R(y) + F(y)]\right] \tag{3.5}$$

Where $G(t)$ is the source term.

We now assume an infinite series solution of the unknown function $y(t)$ of the form:

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{3.6}$$

Then by using Eq. (8), we can re-write Eq. (71) in the form:

$$\sum_{n=0}^{\infty} y_n(t) = G(t) - N^{-1}\left[\frac{u}{s}N^{+}\left[R\sum_{n=0}^{\infty} y_n(t) + \sum_{n=0}^{\infty} A_n(t)\right]\right] \tag{3.7}$$

Where $A_n(t)$ is an Adomian polynomial which represent the nonlinear term. Comparing both sides of Eq. (9), we can easily build the recursive relation as follows:

$$y_0(t) = G(t),$$

$$y_1(t) = -N^{-1}\left[\frac{u}{s}N^{+}[Ry_0(t) + A_0(t)]\right],$$

$$y_2(t) = -N^{-1}\left[\frac{u}{s}N^{+}[Ry_1(t) + A_1(t)]\right],$$

$$y_3(t) = -N^{-1}\left[\frac{u}{s}N^{+}[Ry_2(t) + A_2(t)]\right],$$

Finally, we have the general recursive relation as follows:

$$y_{n+1}(t) = -N^{-1}\left[\frac{u}{s}N^{+}[Ry_n(t) + A_n(t)]\right], n \geq 0$$

Hence, the exact or approximate solution is given by

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

Operated Examples

Example 1. Consider the following second order nonlinear Differential equations

$$y''(x) + (y'(x))^2 + y^2(x) = 1 - cosx, \quad (4.1)$$

Subject to the initial condition

$$y(0) = 1, y'(0) = 0 \quad (4.2)$$

Solution:

By taking N-transform to both sides of equation (4.1) yields,

$$\frac{s^2}{u^2}Y(s,u) - \frac{s}{u^2}y(0) - \frac{1}{u}y'(0) + N^{+}\left[(y'(x))^2\right] + N^{+}[y^2(x)] = \frac{1}{s} - \frac{s}{s^2+u^2} \quad (4.3)$$

By substituting Eq. (4.2) into Eq. (4.3) yields,

$$\frac{s^2}{u^2}Y(s,u) - \frac{s}{u^2} + N^{+}\left[(y'(x))^2\right] + N^{+}[y^2(x)] = \frac{1}{s} - \frac{s}{s^2+u^2} \quad (4.4)$$

$$Y(s,u) = \frac{u^2}{s^3} + \frac{s}{s^2+u^2} - \frac{u^2}{s^2}N^{+}\left[(y'(x))^2 + y^2(x)\right] \quad (4.5)$$

By applying the inverse N-transform, we have,

$$y(x) = N^{-1}\left[\frac{u^2}{s^3} + \frac{s}{s^2+u^2}\right] - N^{-1}\left[\frac{u^2}{s^2}N^{+}\left[(y'(x))^2 + y^2(x)\right]\right] \quad (4.6)$$

$$y(x) = \frac{x^2}{2} + cosx - N^{-1}\left[\frac{u^2}{s^2}N^{+}\left[(y'(x))^2 + y^2(x)\right]\right] \quad (4.7)$$

Now using an infinite series solution of the unknown function of the form,

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \quad (4.8)$$

By using Eq. (4.8), we can re-write Eq. (4.7) as follows:

$$\sum_{n=0}^{\infty} y_n(x) = \frac{x^2}{2} + cosx - N^{-1}\left[\frac{u^2}{s^2}N^{+}\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n\right]\right] \quad (4.9)$$

Where A_n and B_n are the Adomian polynomials of the nonlinear terms $\left(y'(x)\right)^2$ and $y^2(x)$ respectively.
Then by comparing both sides of Equation (4.9), we can drive the general recursive relation as follows,

$$y_0(x) = \frac{x^2}{2} + cosx\,,$$

$$y_1(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_0 + B_0]\right],$$

$$y_2(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_1 + B_1]\right],$$

$$y_3(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_2 + B_2]\right],$$

Hence, the general recursive relation is given by,

$$y_{n+1}(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_n + B_n]\ \ ,n \geq 0\right] \tag{4.10}$$

Then by using the recursive relation derived in Eq. (4.10), we can compute the remaining components of the unknown function $y(x)$ as follows,

$$y_1(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_0 + B_0]\right]$$

$$= -N^{-1}\left[\frac{u^2}{s^2}N^+\left[\left(y_0'(0)\right)^2 + \left(y_0(0)\right)^2\right]\right]$$

$$= -N^{-1}\left[\frac{u^2}{s^2}N^+[1]\right]$$

$$= -N^{-1}\left[\frac{u^2}{s^2}\right]$$

$$= -\frac{x^2}{2}$$

$$y_2(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_1 + B_1]\right]$$

$$= -N^{-1}\left[\frac{u^2}{s^2}N^+\left[\left(y_1'(0)\right)^2 + \left(y_1(0)\right)^2\right]\right]$$

$$= -N^{-1}\left[\frac{u^2}{s^2}N^+[0]\right]$$

$$= -N^{-1}[0]$$
$$= 0$$

Similarly, $y_3(x) = y_4(x) = \cdots = 0$

The approximate solution leads to

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots = \sum_{n=0}^{\infty} y_n(x)$$

Therefore $y(x) = cosx$

Conclusion

In this paper, we successfully found exact solution using NDM to all three nonlinear second order ordinary differential equations. The NDM introduces significant improvement in the fields over existing technique. The NDM introduces a significant improvement in the fields over existing techniques. Our goal in the future is to apply the NDM to other nonlinear second order ordinary differential equations that arise in other areas of science and engineering.

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