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Permanence and uniform asymptotic stability of almost periodic positive solutions for COVID-19 epidemic model with time delays

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Abstract
In this paper we study a non-autonomous time-delayed COVID-19 epidemic model. By utilizing some differential inequalities, sufficient conditions are derived for the permanence of the model and we also obtain the existence and uniform asymptotic stability of almost periodic solutions for the addressed model by Lyapunov functional method. Finally numerical simulations are given to demonstrate our theoretical results.

Keywords: COVID-19, uniform asymptotic stability, differential inequalities, Lyapunov functional method

Introduction

Wuhan, China is the origin of COVID-19 and one of the Cities most affected by it. The Mayor of Wuhan stated at a press conference on January 31,2020 that Wuhan is urgently building Vulcan Mountain Hospital and Thunder Mountain Hospital patients will be officially admitted on February 3 and February 6 [5]. By 24:00 on February 6, 2020, a total of 31,161 confirmed cases, including 636 deaths, were reported in the Chinese mainland, 22,112 confirmed cases, including 618 deaths, were reported in Hubei province, and 11,618 confirmed cases, including 478 deaths, and were reported in Wuhan city. The spread of COVID-19 and various interventions have had an incalculable negative impact on People's daily lives and the normal functioning of society. Cities in China's Hubei Province have issued varying degrees of closures and traffic restrictions [6]. In fact, there are many imminent questions about the spread of COVID-19. How many people will be infected tomorrow? When will the inflection point of the infection rate appear? How many people will be infected during the peak period? Can existing interventions effectively control the COVID-19? What mathematical models are available to help us answer these questions?

The COVID-19 is a novel coronavirus that was only discovered in December 2019, so data on the outbreak is still insufficient, and medical means such as clinical trials are still in a difficult exploratory stage [15]. So far, epidemic data have been difficult to apply directly to existing mathematical models, and questions need to be addressed as to how effective the existing emergency response has been and how to invest medical resources more scientifically in the future and so on. Based on this, this article aims to study the gaps in this part.

Several factors complicate the infection dynamics of COVID-19 and add challenges to the disease control. First, the origin of the infection is still uncertain, although it is widely speculated that wild animals such as bats, civets and minks are responsible for starting the epidemic [25]. Second, clinical evidence shows that the incubation period of this disease ranges from 2 to 14 days. During this period of time, infected individuals may not develop any symptoms and may not be aware of their infection, yet they are capable of transmitting the disease to other people [18]. Third, the virus is new and there are no antiviral drugs or vaccines currently available. Consequently, disease control heavily relies on prompt detection and isolation of symptomatic cases. A number of modeling studies have already been performed for the COVID-19 epidemic. In [22], Wu *et al.*, proposed SEIR model to describe the transmission dynamics, and estimated that the basic reproductive number for COVID-19 was about 2.68. In [17] Read *et al*. reported a value of 3.1 for the basic reproductive number based on data fitting of a SEIR model, using an assumption of poisson-distributed daily time increments.

In [20], Tang *et al*., proposed a deterministic compartmental model incorporating the clinical progression of the disease, the individual epidemiological status, and the intervention measures and estimated the reproductive number could be as high as 6.47, and reported that the quarantine and isolation can effectively reduce the control reproduction number and the transmission risk.

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In nature, the variation of the environment plays an important role in many biological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters in the system incorporates the periodicity of the environment. In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits and harvesting. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic, since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic and more general when we consider the effects of the environmental factors. In fact, several different periodic models have been studied for example (see $[8-0,12,14]$ and references therein).

Mathematical Formulation of the Model

We divide the total human population into five compartments: the susceptible S, the exposed E (individuals in this class are in the incubation period; they do not show symptoms but are still capable of infecting others), the infected I (individuals in this class have fully developed disease symptoms and can infect other people), the quarantined Q, the recovered R.

Motivated by above mentioned works, we introduce the following model to describe the transmission dynamics of the COVID-19 epidemic.

$$\left.\begin{aligned}
S'(t) &= \pi(t) - \alpha(t)S(t)E(t) - [\beta(t)+\gamma(t)+\delta(t)]S(t)\\
E'(t) &= \alpha(t)S(t)E(t) - \sigma(t)E(t-\tau) - [n(t)+\delta(t)]E(t)\\
Q'(t) &= \beta(t)S(t) + \eta(t)E(t) - [\rho(t)+\mu(t)+\delta(t)]Q(t)\\
I'(t) &= \sigma(t)E(t-\tau) + \rho(t)Q(t) - [K(t)+\xi(t)+\delta(t)]I(t)\\
R'(t) &= \gamma(t)S(t) + \mu(t)Q(t) + K(t)I(t) - \delta(t)R(t)
\end{aligned}\right\}\cdots(v19)$$

The parameters of the model are described in Table 1 and they are assumed to be positive and τ is the latent delay of the disease(i.e., the time-delay effect applied to the infected people meaning that the exposed people do not immediately become infected people at any time). A schematic representation of the model (V_{19}) is shown in the following Fig 1.

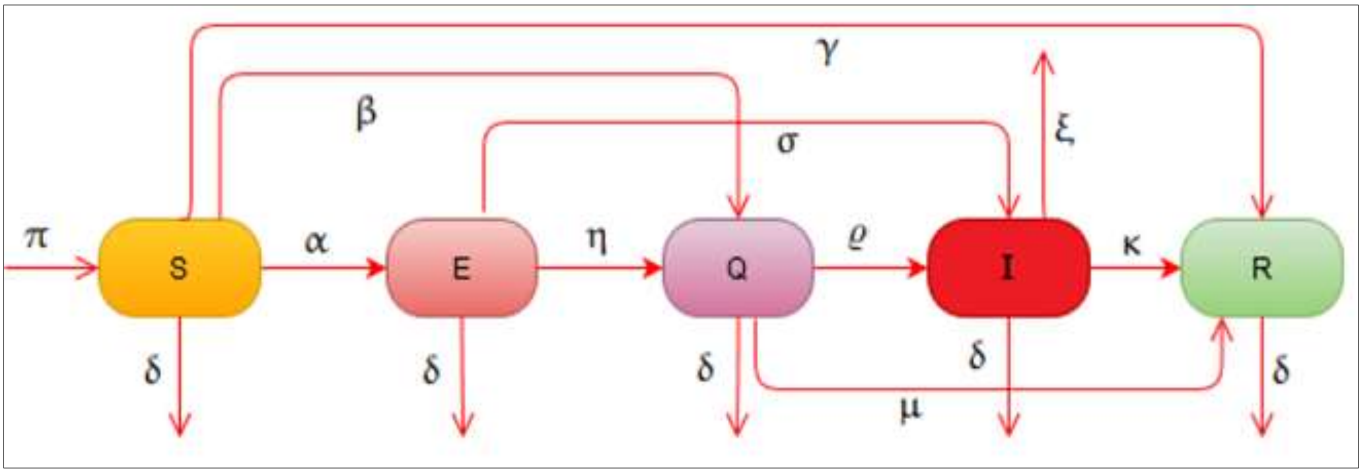

Fig 1: Flow chart of V_19

Preliminaries

In this section, we discuss the permanence of the system $(\mathrm{V}_{19}) - (\mathrm{C}_{19})$, and demonstrate how the disease will be permanent under some conditions.

Table 1: Parameter description and estimates for COVID-19

Parameter symbols	Parameter description
π	the population influx
δ	natural death rate of population
α	transmission rate from susceptible population to exposed population
β	transmission rate from susceptible population to quarantine population
γ	transmission rate from susceptible population to recovered population
σ	transmission rate from exposed population to infected population
η	transmission rate from exposed population to quarantine population
μ	transmission rate from quarantine population to recovered population
ϱ	transmission rate from quarantine population to infected population
κ	transmission rate from infected population to recovered population
ξ	death rate of infected population due to COVID19 virus

For biological reasons, the initial conditions are nonnegative continuous functions as

$$\left.\begin{array}{l}\mathrm{S}(\theta)=\varphi_1(\theta)>0, \mathrm{E}(\theta)=\varphi_2(\theta)>0, \mathrm{Q}(\theta)=\varphi_3(\theta)>0 \\ \mathrm{I}(\theta)=\varphi_4(\theta)>0, \mathrm{R}(\theta)=\varphi_5(\theta)>0, \theta \in[-\tau, 0)\end{array}\right\} \cdots (C19)$$

Where $\varphi=(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \mathcal{C}$ such that $\varphi_i(\theta)>0, (i=1,2,3,4,5)$ for all $\theta \in[-\tau, 0)$ and $\mathcal{C}$ denotes the Banach space $\mathcal{C}([-\tau, 0], \mathbb{R}^5)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^5$ equipped with the norm $\| \varphi \|=\sup_{\theta \in[-\tau, 0)}\{|\varphi_i(\theta)|: i=1,2,3,4,5\}$. Also assume that $\varphi_i(0)>0$ for $i=1,2,3,4,5$

Throughout the paper we denote $f^u=\sup_{t \in \mathbb{R}^+} f(t)$ and $f^\ell=\inf_{t \in \mathbb{R}^+} f(t)$ for an almost periodic function $f(t)$ defined on $\mathbb{R}^+$. Further, we assume that $(\mathcal{H}_1)$ $\pi(t), \alpha(t), \beta(t), \gamma(t), \delta(t), \eta(t), \sigma(t), \rho(t), \mu(t), \kappa(t), \xi(t)$ are all bounded non negative almost periodic functions on $\mathbb{R}^+$ such that.

$$\begin{aligned}
&0<\pi^\ell \leq \pi(t) \leq \pi^u, 0<\alpha^\ell \leq \alpha(t) \leq \alpha^u, 0<\beta^\ell \leq \beta(t) \leq \beta^u, 0<\gamma^\ell \leq \gamma(t) \leq \gamma^u \\
&0<\delta^\ell \leq \delta(t) \leq \delta^u, 0<\eta^\ell \leq \eta(t) \leq \eta^u, 0<\sigma^\ell \leq \sigma(t) \leq \sigma^u, 0<\rho^\ell \leq \rho(t) \leq \rho^u \\
&0<\mu^\ell \leq \mu(t) \leq \mu^u, 0<\kappa^\ell \leq \kappa(t) \leq \kappa^u, 0<\xi^\ell \leq \xi(t) \leq \xi^u
\end{aligned}$$

Definition 3.1. Let $u: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous in t. $u(t)$ is said to be almost periodic on R if, for any $\varepsilon>0$, the set $T(u, \varepsilon)=\{\tau:|u(t+\tau)-u(t)|<\varepsilon$ for all $t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon>0$, it is possible to find a real number $l=l(\varepsilon)>0$, for any interval with length $l(\varepsilon)$, there exists a number $\tau=\tau(\varepsilon)$ in this interval such that $|u(t+\tau)-u(t)|<\varepsilon$ for all $t \in \mathbb{R}$.

Definition 3.2. The system (V_{19}) is said to be permanent if there are positive constants m_i and $M_i (i=1,2,3,4,5)$ such that.

$$\begin{aligned}
&m_1 \leq \liminf_{t \rightarrow+\infty} \mathrm{S}(t) \leq \limsup_{t \rightarrow+\infty} \mathrm{S}(t) \leq M_1, m_2 \leq \liminf_{t \rightarrow+\infty} \mathrm{E}(t) \leq \limsup_{t \rightarrow+\infty} \mathrm{E}(t) \leq M_2 \\
&m_3 \leq \liminf_{t \rightarrow+\infty} \mathrm{Q}(t) \leq \limsup_{t \rightarrow+\infty} \mathrm{Q}(t) \leq M_3, m_4 \leq \liminf_{t \rightarrow+\infty} \mathrm{I}(t) \leq \limsup_{t \rightarrow+\infty} \mathrm{I}(t) \leq M_4 \\
&m_5 \leq \liminf_{t \rightarrow+\infty} \mathrm{R}(t) \leq \limsup_{t \rightarrow+\infty} \mathrm{R}(t) \leq M_5
\end{aligned}$$

Hold for any solution $(\mathrm{s}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t))$ of (v_{19}) with initial conditions (C_{19}). Here m_i and $M_i (i=1,2,3,4,5)$ are independent of initial conditions (C_{19}) Lemma 3.3. Assume that $(\mathrm{S}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t))$ is any solution of system (V_{19}) with the initial conditions (C_{19}), then $\mathrm{S}(t)>0, \mathrm{E}(t)>0, \mathrm{Q}(t)>0, \mathrm{I}(t)>0, \mathrm{R}(t)>0$ for all $t \in \mathbb{R}^+$

Proof. since the right hand side of (V_{19}) is completely continuous and locally Lipschitzian on C, the solution $(\mathrm{S}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t))$ of (V_{19}) with initial conditions (C_{19}) exists, and is unique on $[0, t_0)$, for some $0<t_0<+\infty$. From the first equation of (V_{19}), we have.

$$\begin{aligned}
&\mathrm{S}'(t) \geq-(\beta^u+\gamma^u+\delta^u) \mathrm{S}(t) \\
\Rightarrow \quad &\mathrm{S}(t) \geq \varphi_1(0) \exp\{-(\beta^u+\gamma^u+\delta^u) t\}>0
\end{aligned}$$

From the second equation of (V_{19}), we get

$$\mathrm{E}'(t) \geq -(\sigma(t) + \eta(t) + \delta(t))\mathrm{E}(t)$$

$$\Rightarrow \mathrm{E}(t) \geq \varphi_2(0)\exp\{-(\sigma^u + \eta^u + \delta^u)t\} > 0$$

Similarly, other equations of (V_{19}), we obtained

$$\mathrm{Q}(t) = \varphi_3(0)\exp\{-(\rho^u + \mu^u + \delta^u)t\} > 0$$
$$\mathrm{I}(t) = \varphi_4(0)\exp\{-(\kappa^u + \xi^u + \delta^u)t\} > 0$$
$$\mathrm{R}(t) = \varphi_3(0)\exp\{-\delta^u t\} > 0$$

This completes the proof

Lemma 3.4. Let $(\mathrm{S}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t))$ be any solution of (V_{19}) with initial conditions (C_{19}) and let $\mathrm{N}(t) = \mathrm{S}(t) + \mathrm{E}(t) + \mathrm{Q}(t) + \mathrm{I}(t) + \mathrm{R}(t)$ for $t \geq 0$. Then.

$$\limsup_{t\to+\infty} \mathrm{N}(t) \leq \frac{\pi^u}{\delta^\ell}$$

Proof. From Lemma 3.3, $\mathrm{S}(t) > 0, \mathrm{E}(t) > 0, \mathrm{Q}(t) > 0, \mathrm{I}(t) > 0, \mathrm{R}(t) > 0$ for all $t \in [0, t_0)$ where $t_0 \in \mathbb{R}^+$. Thus for $t \in [0, t_0)$ and adding all equations of (V_{19}), we get

$$\begin{aligned} \mathrm{N}'(t) &= \pi(t) - \delta(t)\mathrm{N}(t) \\ &\leq \pi^u - \delta^\ell \mathrm{N}(t) \end{aligned}$$

Which implies that

$$\limsup_{t\to+\infty} \mathrm{N}(t) \leq \frac{\pi^u}{\delta^\ell}$$

That is $\big(\mathrm{S}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t)\big)$ is uniformly bounded on $[0, t_0)$. From [11], we have $t_0 = +\infty$.

This completes the proof.

Lemma 3.5. Suppose $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ $\pi^\ell \geq (\beta^u + \gamma^u + \eta^u + \sigma^u)\left(\frac{\pi^u}{\delta^\ell}\right)$ hold and let $(\mathrm{S}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t))$ be any solution of (V_{19}) with initial conditions (C_{19}). Then there exist $m_i > 0, i = 1,2,3,4,5$ such that

$$\liminf_{t\to+\infty} \mathrm{S}(t) \geq m_1, \liminf_{t\to+\infty} \mathrm{E}(t) \geq m_2, \liminf_{t\to+\infty} Q(t) \geq m_3$$
$$\liminf_{t\to+\infty} \mathrm{I}(t) \geq m_4, \liminf_{t\to+\infty} \mathrm{R}(t) \geq m_5$$

Proof. By Lemma 3.4, for any $\varepsilon > 0$, there is a large sufficiently $t_1 > 0$ such that

$$\mathrm{E}(t) \leq \frac{\pi^u}{\delta^\ell} + \varepsilon$$

As $t \geq t_1$. Thus, from the first equation of system (V_{19}), when $t \geq t_1$

$$\begin{aligned} \mathrm{S}'(t) &\geq \pi(t) - \left[\alpha(t)\left(\frac{\pi^u}{\delta^\ell} + \varepsilon\right) + \beta(t) + \gamma(t) + \delta(t)\right]\mathrm{S}(t) \\ &\geq \pi^\ell - \left[\alpha^u\left(\frac{\pi^u}{\delta^\ell} + \varepsilon\right) + \beta^u + \gamma^u + \delta^u\right]\mathrm{S}(t) \end{aligned}$$

Which implies that

$$\liminf_{t\to+\infty} \mathrm{S}(t) \geq \frac{\pi^\ell}{\alpha^u\left(\frac{\pi^u}{\delta^\ell} + \varepsilon\right) + \beta^u + \gamma^u + \delta^u}$$

Since ε is arbitrarily small, it follows that

$$\liminf_{t\to+\infty} \mathrm{S}(t) \geq \pi^\ell\left[\alpha^u\left(\frac{\pi^u}{\delta^\ell}\right) + \beta^u + \gamma^u + \delta^u\right]^{-1} := m_1$$

Next, letting $\mathrm{P}(t) = \mathrm{S}(t) + \mathrm{E}(t)$ and adding first two equations of (V_{19}), We get

$$\begin{aligned}\mathrm{P}'(t) &= \pi(t)-\sigma(t)\mathrm{E}(t-\tau)-[\beta(t)+\gamma(t)]\mathrm{S}(t)-\mathrm{n}(t)\mathrm{E}(t)-\delta(t)\mathrm{P}(t)\\ &\geq \pi^{\ell}-\sigma^{u}\left(\frac{\pi^{u}}{\delta^{\ell}}+\varepsilon\right)-(\beta^{u}+\gamma^{u})\left(\frac{\pi^{u}}{\delta^{\ell}}+\varepsilon\right)-\eta^{u}\left(\frac{\pi^{u}}{\delta^{\ell}}+\varepsilon\right)-\delta^{u}\mathrm{P}(t)\end{aligned}$$

Which implies that

$$\liminf_{t\to+\infty}\mathrm{P}(t)\geq\frac{1}{\delta^{u}}\left[\pi^{\ell}-(\sigma^{u}+\beta^{u}+\gamma^{u}+\eta^{u})\left(\frac{\pi^{u}}{\delta^{\ell}}+\varepsilon\right)\right]$$

From the definition of $P(t)$, we have

$$\liminf_{t\to+\infty}\mathrm{E}(t)\geq\frac{1}{\delta^{u}}\left[\pi^{\ell}-(\sigma^{u}+\beta^{u}+\gamma^{u}+\eta^{u})\left(\frac{\pi^{u}}{\delta^{\ell}}\right)\right]-m_1:=m_2$$

Similarly, we can have

$$\begin{aligned}\liminf_{t\to+\infty}Q(t)&\geq\frac{\beta^{\ell}m_1+\eta^{\ell}m_2}{\rho^{u}+\mu^{u}+\delta^{u}}:=m_3\\ \liminf_{t\to+\infty}I(t)&\geq\frac{\sigma^{\ell}m_2+\rho^{\ell}m_3}{\kappa^{u}+\xi^{u}+\delta^{u}}:=m_4\\ \liminf_{t\to+\infty}R(t)&\geq\frac{\gamma^{\ell}m_1+\mu m_3+\kappa^{\ell}m_4}{\delta^{u}}:=m_5\end{aligned}$$

This completes the proof.

Theorem 3.6. Assume $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Then the system (V_{19}) with initial conditions (C_{19}) is permanent.

Proof. From Lemmas 3.4 and 3.5, the system (V_{19}) is permanent.

Uniform asymptotic stability of positive almost periodic solutions

In this section, we establish sufficient conditions for the existence, uniqueness and uniform asymptotic stability of positive almost periodic solution of system (V_{19}) and (C_{19}). In this regard we utilize the following theorem:

Consider the following almost periodic system.

$$\mathrm{x}'(t)=\mathrm{g}(t,\mathrm{x}(t))\cdots\cdots(1)$$

and its associate product system

$$\mathbf{x}'(t)=\mathbf{g}(t,\mathbf{x}(t)),\mathbf{y}'(t)=\mathbf{g}(t,\mathbf{y}(t))\cdots\cdots(2)$$

Where $\mathrm{g}:\mathbb{R}\times\mathcal{B}_{\mathrm{M}}\to\mathbb{R},\mathcal{B}_{\mathrm{M}}=\{\mathrm{z}\in\mathbb{R}^{n}:\|\mathrm{z}\|<\mathrm{M}\},\|\mathrm{z}\|=\sup_{t\in\mathbb{R}}|\mathbf{z}(t)|,\mathrm{g}(t,\mathrm{z})$ is almost

periodic in t uniformly for $z\in\mathcal{B}_M$ and is continuous in z.

Then we have following result.

Lemma 4.1([24]). Let $\mathcal{V}(t,u,v)$ be Lyapunov function defined on $\mathbb{R}^{+}\times\mathcal{S}_{\mathrm{M}}\times\mathcal{S}_{\mathrm{M}}$ and satisfies the Following conditions

(i) $\mathrm{A}(\|u-v\|)\leq\mathcal{V}(t,u,v)\leq\mathrm{B}(\|u-v\|)$, where $\mathrm{A},\mathrm{B}\in\mathcal{P}$

$$\mathcal{P}=\{\mathrm{G}\in\mathcal{C}(\mathbb{R}^{+},\mathbb{R}^{+}):\mathrm{G}(0)=0\mathrm{andGisincreasing}\}$$

(ii) $|\mathcal{V}(t,u_1,v_1)-\mathcal{V}(t,u_2,v_2)|\leq\mathcal{L}(\|u_1-u_2\|+\|v_1-v_2\|)$, where $\mathcal{L}>0$ is a constant,

(iii) $\mathfrak{D}^{+}\mathcal{V}(t,u,v)|_{(2)}\leq-c\mathcal{V}(t,u,v)$, where $c>0$.

Further, assume that there a solution $\mathrm{z}(t)\in\mathcal{S}_{\mathrm{M}}$ of system (2).

Then there exist a unique almost periodic solution $\mathrm{w}(t)\in\mathcal{S}$ of system (2), which is uniformly asymptotically stable.

Define

$$\begin{aligned}&\Pi=\{(\mathrm{S}(t),\mathrm{E}(t),Q(t),\mathrm{I}(t),\mathrm{R}(t))^{T}\in\mathbb{R}^{+5}:0<m_1\leq\mathrm{S}(t)\leq M_1,0<m_2\leq\mathrm{E}(t)\leq M_2\\ &0<m_3\leq Q(t)\leq M_3,0<m_4\leq\mathrm{I}(t)\leq M_4,0<m_5\leq\mathrm{R}(t)\leq M_5\}\end{aligned}$$

Then, it is clear that $\Pi\neq\emptyset$ and is invariant set of system (V_{19}). Theorem 4.2. Assume $(\mathcal{H}_1),(\mathcal{H}_2)$ and the following holds. $(\mathcal{H}_3)$ there is some $c>0$ such that $c=\min\{c_1,c_2,c_3,c_4,\delta^{\ell}\}$ where

$$c_1 = (1+m_2)\alpha^\ell + \beta^\ell + \gamma^\ell + \delta^\ell - \beta^u - \gamma^u > 0, c_2 = 2\alpha^\ell m_1 + \eta^\ell + \delta^\ell - \sigma^u - \eta^u > 0$$
$$c_3 = \rho^\ell + \mu^\ell + \delta^\ell - \rho^u - \mu^u > 0, c_4 = \kappa^\ell + \xi^\ell + \delta^\ell - \kappa^u > 0$$

Then the dynamic system (V_{19}) with initial conditions (C_{19}) has a unique almost periodic positive solution $(\mathrm{s}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t)) \in \Pi$ and is uniformly asymptotically stable.
Proof. According to Theorem 3.6, every solution $(\mathrm{s}(t), \mathrm{E}(t), Q(t), \mathrm{I}(t), \mathrm{R}(t))$ of system (V_{19}) satisfies that $m_1 \leq \mathrm{S}(t) \leq M_1, m_2 \leq \mathrm{E}(t) \leq M_2, m_3 \leq \mathrm{Q}(t) \leq M_3, m_4 \leq \mathrm{I}(t) \leq M_4, m_5 \leq \mathrm{R}(t) \leq M_5$. Hence, $|\mathrm{S}(t)| \leq A, |\mathrm{E}(t)| \leq B, |\mathrm{Q}(t)| \leq C, |\mathrm{I}(t)| \leq D, |\mathrm{R}(t)| \leq E$ where$A_i = \max\{|m_1|, |M_1|\}, B = \max\{|m_2|, |M_2|\}, C = \max\{|m_3|, |M_3|\}, D = \max\{|m_4|, |M_4|\}$ and $E = \max\{|m_5|, |M_5|\}$ Denote

$$\| (\mathrm{S}(t), \mathrm{E}(t), Q(t), \mathrm{I}(t), \mathrm{R}(t)) \| = \sup_{t\in\mathbb{R}^+} |\mathrm{S}(t)| + \sup_{t\in\mathbb{R}^+} |\mathrm{E}(t)| + \sup_{t\in\mathbb{R}^+} |Q(t)| + \sup_{t\in\mathbb{R}^+} |\mathrm{I}(t)| + \sup_{t\in\mathbb{R}^+} |\mathrm{R}(t)|$$

Suppose that $X = (\mathrm{S}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t)), \hat{X} = (\hat{\mathrm{S}}(t), \hat{\mathrm{E}}(t), \hat{\mathrm{Q}}(t), \hat{\mathrm{I}}(t), \hat{\mathrm{R}}(t))$ are any two positive solutions of system (V_{19}), then

$$\| X \| \leq A+B+C+D+E \text{ and } \| \hat{X} \| \leq A+B+C+D+E$$

The product system of (V_{19}) reads

$$\left.\begin{array}{ll}
\mathrm{S}'(t) & = \pi(t) - \alpha(t)\mathrm{S}(t)\mathrm{E}(t) - [\beta(t)+\gamma(t)+\delta(t)]\mathrm{S}(t) \\
\mathrm{E}'(t) & = \alpha(t)\mathrm{S}(t)\mathrm{E}(t) - \sigma(t)\mathrm{E}(t-\tau) - [\mathrm{n}(t)+\delta(t)]\mathrm{E}(t) \\
\mathrm{Q}'(t) & = \beta(t)\mathrm{S}(t) + \eta(t)\mathrm{E}(t) - [\rho(t)+\mu(t)+\delta(t)]\mathrm{Q}(t) \\
\mathrm{I}'(t) & = \sigma(t)\mathrm{E}(t-\tau) + \rho(t)\mathrm{Q}(t) - [\mathrm{K}(t)+\xi(t)+\delta(t)]\mathrm{I}(t) \\
\mathrm{R}'(t) & = \gamma(t)\mathrm{S}(t) + \mu(t)\mathrm{Q}(t) + \mathrm{K}(t)\mathrm{I}(t) - \delta(t)\mathrm{R}(t) \\
\hat{\mathrm{S}}'(t) & = \pi(t) - \alpha(t)\hat{\mathrm{S}}(t)\hat{\mathrm{E}}(t) - [\beta(t)+\gamma(t)+\delta(t)]\hat{\mathrm{S}}(t) \\
\hat{\mathrm{E}}'(t) & = \alpha(t)\hat{\mathrm{S}}(t)\hat{\mathrm{E}}(t) - \sigma(t)\hat{\mathrm{E}}(t-\tau) - [\mathrm{n}(t)+\delta(t)]\hat{\mathrm{E}}(t) \\
\hat{\mathrm{Q}}'(t) & = \beta(t)\hat{\mathrm{S}}(t) + \eta(t)\hat{\mathrm{E}}(t) - [\rho(t)+\mu(t)+\delta(t)]\hat{\mathrm{Q}}(t) \\
\hat{\mathrm{I}}'(t) & = \sigma(t)\hat{\mathrm{E}}(t-\tau) + \rho(t)\hat{\mathrm{Q}}(t) - [\mathrm{K}(t)+\xi(t)+\delta(t)]\hat{\mathrm{I}}(t) \\
\hat{\mathrm{R}}'(t) & = \gamma(t)\hat{\mathrm{S}}(t) + \mu(t)\hat{\mathrm{Q}}(t) + \mathrm{K}(t)\hat{\mathrm{I}}(t) - \delta(t)\hat{\mathrm{R}}(t)
\end{array}\right\} \cdots (3)$$

Define the Lyapunov function $\mathcal{V}(t, X, \hat{X})$ on $\mathbb{R}^+ \times \Pi \times \Pi$ as

$$\mathcal{V}(t, X, \hat{X}) = |\mathrm{S}(t) - \hat{\mathrm{S}}(t)| + |\mathrm{E}(t) - \hat{\mathrm{E}}(t)| + |\mathrm{Q}(t) - \hat{\mathrm{Q}}(t)| + |\mathrm{I}(t) - \hat{\mathrm{I}}(t)| + |\mathrm{R}(t) - \hat{\mathrm{R}}(t)|$$

Define the norm
It is easy to see that there exist two constants $a > 0, b > 0$ such that

$$a \| X(t) - \hat{X}(t) \| \leq V(t, X, \hat{X}) \leq b \| X(t) - \hat{X}(t) \|$$

Let $\mathrm{A}, \mathrm{B} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+), \mathrm{A}(x) = ax, \mathrm{B}(x) = bx$, then the assumption
(i) of Lemma 4.1 is satisfied. On the other hand, we have where $\mathcal{L} = 1$, so condition (ii) of Lemma 4.1 is satisfied. Now consider a function $\mathcal{W}(t) = \mathcal{W}_1(t) + \mathcal{W}_2(t) + \mathcal{W}_3(t) + \mathcal{W}_4(t) + \mathcal{W}_5(t)$, where

$$\mathcal{W}_1(t) = |\mathrm{S}(t) - \hat{\mathrm{S}}(t)|, \mathcal{W}_2(t) = |\mathrm{E}(t) - \hat{\mathrm{E}}(t)| + \sigma^u \int_{t-\tau}^{t} |\mathrm{E}(t) - \hat{\mathrm{E}}(t)| dt$$
$$\mathcal{W}_3(t) = |\mathrm{Q}(t) - \hat{\mathrm{Q}}(t)|, \mathcal{W}_4(t) = |\mathrm{I}(t) - \hat{\mathrm{I}}(t)| \text{and} \mathcal{W}_5(t) = |R(t) - \hat{R}(t)|$$

For $t \in \mathbb{R}^+$, calculating the Dini derivative $\mathfrak{D}^+\mathcal{W}_1(t)$ of $\mathcal{W}_1(t)$ along system (V_{19}), we get $\mathfrak{D}^+\mathcal{W}_1(t) \leq sign(\mathrm{S}(t) - \hat{\mathrm{S}}(t))[\mathrm{S}(t) - \hat{\mathrm{S}}(t)]'$

$$\begin{aligned}
&\leq sign(\mathrm{S}(t) - \hat{\mathrm{S}}(t))[\alpha(t)\hat{\mathrm{S}}(t)\hat{\mathrm{E}}(t) - \alpha(t)\mathrm{S}(t)\mathrm{E}(t) - (\beta(t)+\gamma(t)+\delta(t))(\mathrm{S}(t) - \hat{\mathrm{S}}(t))] \\
&\leq sign(\mathrm{S}(t) - \hat{\mathrm{S}}(t))[-\alpha(t)\mathrm{E}(t)(\mathrm{S}(t) - \hat{\mathrm{S}}(t)) - \alpha(t)\hat{\mathrm{S}}(t)(\mathrm{E}(t) - \hat{\mathrm{E}}(t)) \\
&\quad -(\beta(t)+\gamma(t)+\delta(t))(\mathrm{S}(t) - \hat{\mathrm{S}}(t))] \\
&\leq -\left(\alpha^\ell + \beta^\ell + \gamma^\ell + \delta^\ell\right)|\mathrm{S}(t) - \hat{\mathrm{S}}(t)| - \alpha^\ell m_1 |\mathrm{E}(t) - \hat{\mathrm{E}}(t)|
\end{aligned}$$

and

D^+W_5(t) sign(R(t)-R(t))[R(t)-R(t)]^

l sign(R(t)-R(t))[(t)(S(t)-S(t))+(t)(Q(t)-Q(t))+k(t)(I(t)-I(t))
-(t)(R(t)-R(t))]
^u|S(t)-S(t)|+^u|Q(t)-Q(t)|+K^u|I(t)-I(t)|-^|R(t)-R(t)| and

$$\mathfrak{D}_5^{+\mathcal{W}}(t) \le sign(\mathrm{R}(t) - \widehat{\mathrm{R}}(t))[\mathrm{R}(t) - \widehat{\mathrm{R}}(t)]'$$
$$\le sign(\mathrm{R}(t) - \widehat{\mathrm{R}}(t))[\gamma(t)(\mathrm{S}(t) - \hat{\mathrm{S}}(t)) + \mu(t)(\mathrm{Q}(t) - \widehat{\mathrm{Q}}(t)) + \mathrm{k}(t)(\mathrm{I}(t) - \hat{\mathrm{I}}(t)) - \delta(t)(\mathrm{R}(t) - \widehat{\mathrm{R}}(t))$$
$$\le \gamma^u|\mathrm{S}(t) - \hat{\mathrm{S}}(t)| + \mu^u|Q(t) - \widehat{\mathrm{Q}}(t)| + \mathrm{K}^u|\mathrm{I}(t) - \hat{\mathrm{I}}(t)| - \delta^\ell|\mathrm{R}(t) - \widehat{\mathrm{R}}(t)|$$

But $\mathfrak{D}^+\mathcal{V}(t) \le \mathfrak{D}^+\mathcal{W}(t), t \in \mathbb{R}^+$.
Therefore, from the above inequalities, we get

$$\begin{aligned}
\mathfrak{D}^+\mathcal{V}(t) \le \quad & -\left((1+m_2)\alpha^\ell + \beta^\ell + \gamma^\ell + \delta^\ell - \beta^u - \gamma^u\right)|\mathbf{S}(t) - \hat{\mathrm{S}}(t)| \\
& -\left(2\alpha^\ell m_1 + \eta^\ell + \delta^\ell - \sigma^u - \eta^u\right)|\mathrm{E}(t) - \widehat{\mathrm{E}}(t)| \\
& -\left(\rho^\ell + \mu^\ell + \delta^\ell - \rho^u - \mu^u\right)|Q(t) - \widehat{\mathrm{Q}}(t)| - \left(\kappa^\ell + \xi^\ell + \delta^\ell - \mathrm{k}^u\right)|\mathrm{I}(t) - \hat{\mathrm{I}}(t)| \\
& -\delta^\ell|\mathrm{R}(t) - \widehat{\mathrm{R}}(t)| \\
\le \quad & -c_1|\mathrm{S}(t) - \hat{\mathrm{S}}(t)| - c_2|\mathrm{E}(t) - \widehat{\mathrm{E}}(t)| - c_3|\mathrm{Q}(t) - \widehat{\mathrm{Q}}(t)| \\
& -c_4|\mathrm{I}(t) - \hat{\mathrm{I}}(t)| - \delta^\ell|\mathrm{R}(t) - \widehat{\mathrm{R}}(t)| \\
\le \quad & -c\mathcal{V}(t)
\end{aligned}$$

Thus, the assumption (iii) of Lemaa 4.1 is satisfied and hence, it follows from Lemma 4.1 that there exists a unique uniformly asymptotically stable almost periodic positive solution

$(s(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t))$ of dynamic system (V_{19}) and $(S(t), \mathrm{E}(t), Q(t), \mathrm{I}(t), \mathrm{R}(t)) \in \Pi$

This completes the proof.

Numerical Simulations
Consider the following model

$$\left.\begin{aligned}
\mathrm{S}'(t) &= \pi(t) - \alpha(t)\mathrm{S}(t)\mathrm{E}(t) - [\beta(t) + \gamma(t) + \delta(t)]\mathrm{S}(t) \\
\mathrm{E}'(t) &= \alpha(t)\mathrm{S}(t)\mathrm{E}(t) - \sigma(t)\mathrm{E}(t-\tau) - [\mathrm{n}(t) + \delta(t)]\mathrm{E}(t) \\
\mathrm{Q}'(t) &= \beta(t)\mathrm{S}(t) + \eta(t)\mathrm{E}(t) - [\rho(t) + \mu(t) + \delta(t)]Q(t) \\
\mathrm{I}'(t) &= \sigma(t)\mathrm{E}(t-\tau) + \rho(t)\mathrm{Q}(t) - [\mathrm{K}(t) + \xi(t) + \delta(t)]\mathrm{I}(t) \\
\mathrm{R}'(t) &= \gamma(t)\mathrm{S}(t) + \mu(t)\mathrm{Q}(t) + \mathrm{k}(t)\mathrm{I}(t) - \delta(t)\mathrm{R}(t)
\end{aligned}\right\} \cdots\cdots (4)$$

Where

$$\begin{gathered}
\pi(t) = 1 + \sin(\sqrt{2}t), \alpha(t) = 2 + 0.1\cos(\sqrt{2}t), \beta(t) = 0.05 + 0.05\sin(\sqrt{3}t), \delta(t) = 1 \\
\gamma(t) = 0.05 + 0.05\cos(\sqrt{5}t), \sigma(t) = 0.05 + 0.05\sin(\sqrt{1}t), \mu(t) = 0.02 + 0.01\cos(\sqrt{2}t) \\
\eta(t) = 0.05 + 0.05\sin(\sqrt{3}t), \rho(t) = 0.02 + 0.01\sin(\sqrt{1}t), \kappa(t) = 0.02 + 0.01\cos(\sqrt{2}t) \\
\xi(t) = 0.02 + 0.01\sin(\sqrt{3}t)
\end{gathered}$$

Then

$$\begin{gathered}
m_1 = 0.1851851852, m_2 = 0.0148148148, m_3 = 0.009433962264 \\
m_4 = 0.0008768113069, m_5 = 0.009465474731
\end{gathered}$$

and

$$c_1 = 3.031111112, c_2 = 1.627777778, c_3 = 0.98, c_4 = 1.01$$

So,

$c = \min\{c_1, c_2, c_3, c_4, 1\} = 0.98 > 0$. Thus, all conditions of Theorem 4.2 are satisfied. From Fig. 2-6 it is easy to see that for system (4) there exists a positive almost periodic solution denoted by $(\mathrm{S}^*(t), \mathrm{E}^*(t), \mathrm{Q}^*(t), \mathrm{I}^*(t), \mathrm{R}^*(t))$.

Moreover, Fig. $7-11$ shows that any positive solution $(\mathrm{S}(t), \mathrm{E}(t), \mathrm{Q}(t), \mathrm{I}(t), \mathrm{R}(t))$ tends to the above almost periodic solution

$$(\mathrm{S}^*(t), \mathrm{E}^*(t), \mathrm{Q}^*(t), \mathrm{I}^*(t), \mathrm{R}^*(t))$$

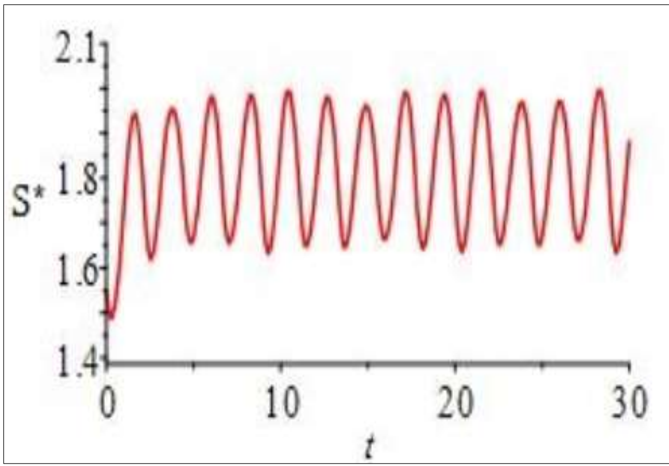

Fig 2: Positive almost periodic solution.(4). Time series of $S^*(t)$ with ini of system.$S^*(0) = 1.55$ and t over [0,30].

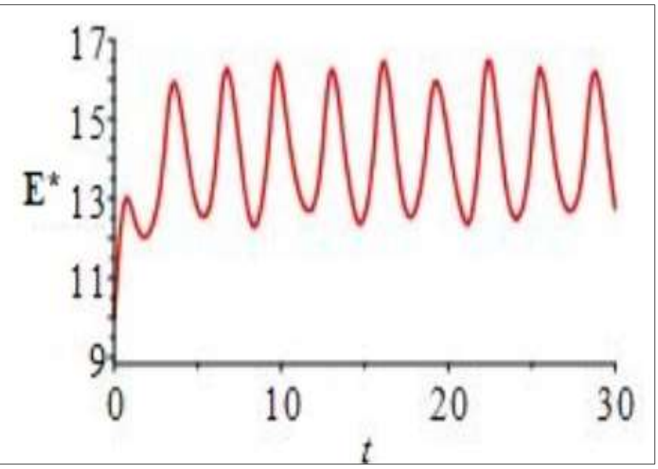

Fig 3: Positive almost periodic solution of system (4). Time series of E^* (t) with initial value E^* (0)=10 and t over [0,30]

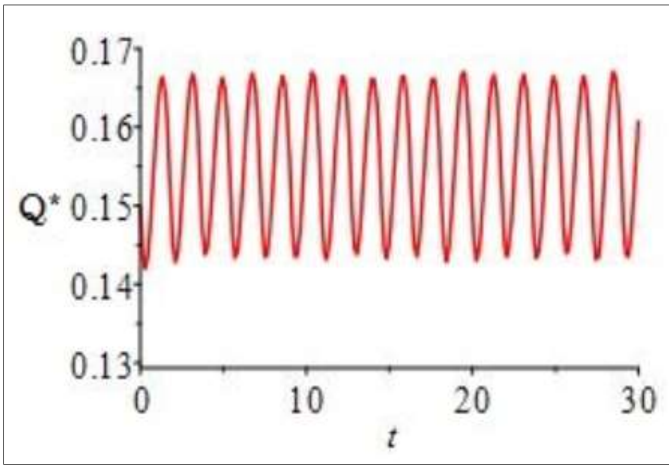

Fig 4: Positive almost periodic solution of system (4).Time series of Q^* (t) with initial value Q^* (0)=0.15 and t over [0,30]

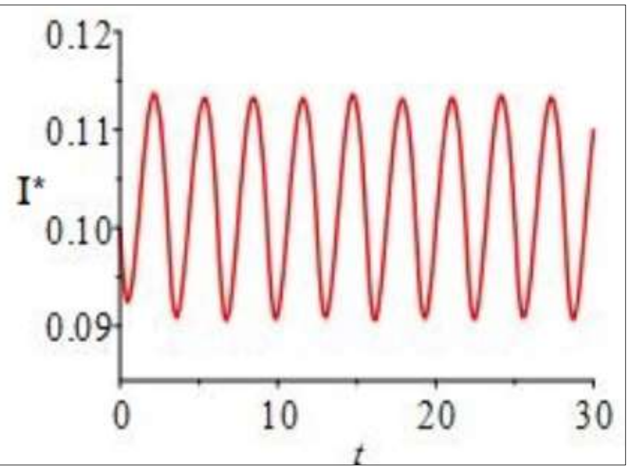

Fig 5: Positive almost periodic solution of system (4) time series

$I^*(t) wihinitialvalue$I^*(0)=0.1$andtover[0,30][width = 17cm, height = 6cm]images/figure\ 5.PNG2Figure6: Positivealmostperiodicsolution(4). Timeseriesof$R^*(t)$withini - system(4)$ $Time\ series\ of$S(t)andS^*(t)$tialvalue$R^*(0)=0.08andt $over[0,30]Figure7: Uniformlyasymptoticstabilityofofsystem(4)\ .timeseriesof$I^*(t) with initial values $S(0) = 1.55, S^*(0) = 1.9$ and t over [0,30]

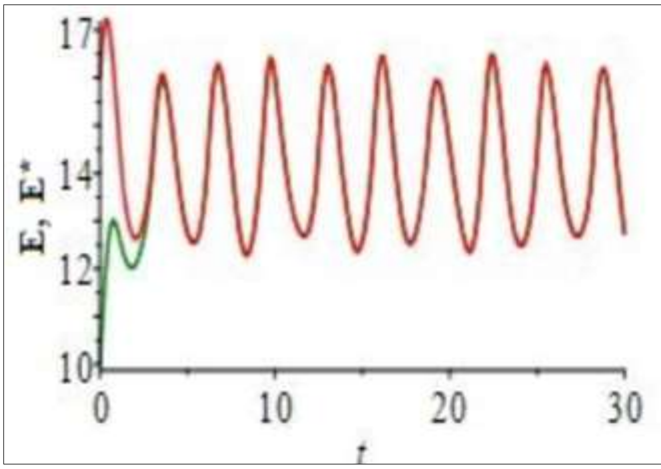

Fig 8: Uniformly asymptotic stability of system (4).Time series of E(t) and E^* (t) with initial values E(0)=10,E^* (0)=16 and t over [0,30]

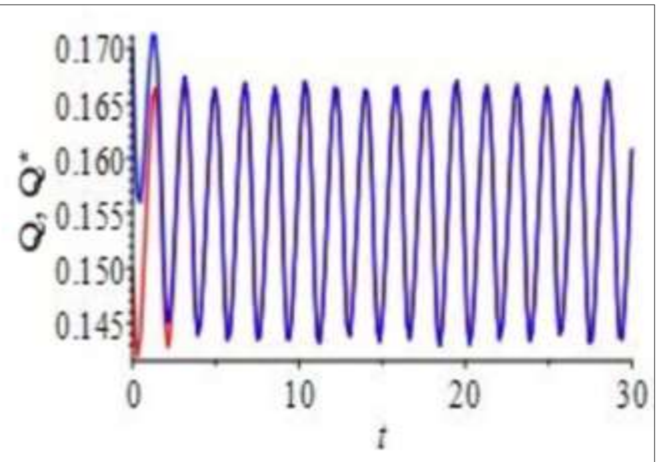

Fig 9: Uniformly asymptotic stability of system (4)Time series of $Q(t)$ and $Q^*(t)$ with initial values $Q(0) = 0.15, Q^*(0) = 0.17$and t over [0,30]

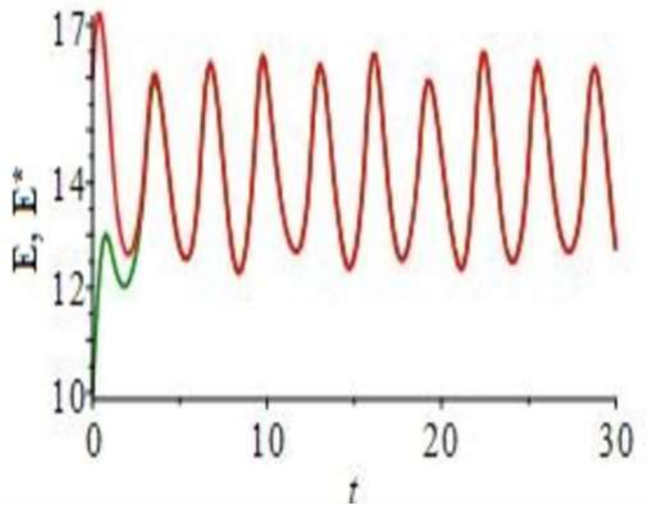

Fig 10: Uniformly asymptotic stability Time series of $\mathrm{R}(t)$ and $\mathrm{R}^*(t)$ with initial values $\mathrm{I}(0) = 0.1, \mathrm{I}^*(0) = 0.11$ with initial values $\mathrm{R}(0) = 0.08, \mathrm{R}^*(0) = 0.12$ and t over $[0,30]$. and t over $[0,30]$

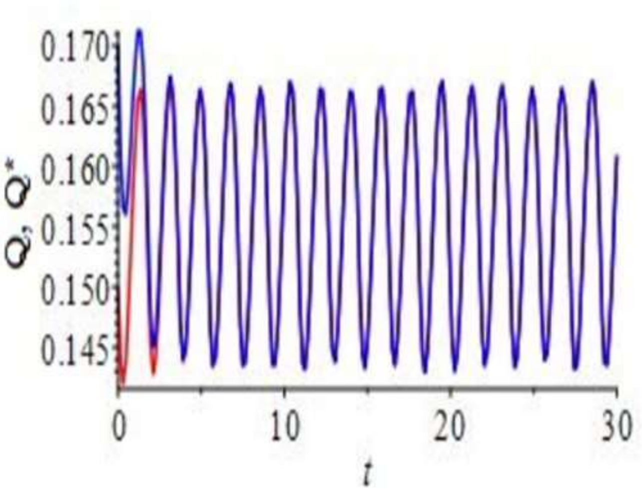

Fig 11: Uniformly asymptotic stability of system (4). Time series of $\mathrm{I}(t)$ and $\mathrm{I}^*(t)$ of system (4).

Conclusion

This paper investigated a non-autonomous time-delayed COVID-19 epidemic model, aiming to address critical questions regarding disease spread and control strategies. By utilizing differential inequalities and the Lyapunov functional method, the study derived conditions for the permanence of the model and demonstrated the existence and uniform asymptotic stability of almost periodic solutions. The analysis provided insights into the long-term behavior of the epidemic dynamics, offering valuable information for policymakers and healthcare professionals. Overall, this research contributes to the understanding of COVID-19 transmission dynamics and provides a mathematical framework for assessing intervention strategies. Further numerical simulations supported the theoretical findings, reinforcing their practical relevance.

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