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A new approach to embedded differential equations: The core-shell approach

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Abstract
A new approach is presented for solving ordinary differential equations. The new approach is named as the Core-Shell approach. Most of the well-established techniques can be re-interpreted and the results can be recovered using this new approach. The analysis is devoted mostly to linear differential equations with constant and variable coefficients. It is hoped that the undergraduate students in applied fields taking differential equation courses will benefit from the new approach by increasing their understanding of the fundamental concepts and solution techniques.

Keywords: Ordinary differential equations, analytical solutions, operator decomposition, core-shell approach

Introduction

Differential equations course is one of the fundamental courses for students looking for a degree in Mathematics, Physics or Engineering. The course is taken at the sophomore year after gaining the necessary skills in Calculus. Students often have hard times in grasping the ideas and fundamentals of the topic and are usually paralysed when encountering a new differential equation to be solved. The method presented in this paper suggests a new unified approach to the understanding of differential equations as well as their solution techniques. The first step is to identify the differential equation as an embedded differential equation, to extract the core and shell parts of it. The next step is to solve the shell equation first and then solve the core equation to finally reach to a solution of the embedded equation. The method resembles much like eating a fruit with a shell covering the core part; one first eliminates the shell and then reaches the core, eats it for satisfying his/her nutrition demands.

First, the theory of the method applicable to the linear and nonlinear equations are given briefly. Then the solutions for constant and variable coefficient second order linear differential equations are derived using the method. Detailed discussions on factorisation of the operators are made. In addition to the approach being a new one, some of the results presented are new and not published before. Where ever applies, proper acknowledgements to the previous work presenting the same results are given. The generalization of the ideas to arbitrary orders of differential equations are given together with some research project suggestions for advanced undergraduate students.

Some of the closely related work on the topic are given as past references and verification of the results presented here obtained by similar and non-similar methods. For constant coefficient second order linear differential equations, Lutzer (2006) [5] presented a general solution of the integral form. Employing the analogy between the first order variable coefficient equations and second order constant coefficient equations and a transformation, Tolle (2011) [10] presented a solution technique for the homogenous constant coefficient second order equations. For constant coefficient linear equations of arbitrary order, Figueroa and Rebolledo (2015) [3] constructed a general solution of the integral form. The work was indeed an extension of their previous work (Figueroa and Rebolledo, 2015a) [4] on second order differential equations which covers constant coefficient as well as variable coefficients. Factorisation method was employed for variable coefficient linear second order equations by Clegg (2006) [2]. By a special transformation based on an exponential integral function, the variable coefficient second order equations with coefficients satisfying certain conditions were solved. Some exact solutions for certain variable coefficient second order equations were also presented (Mohammed and Zeleke, 2015) [6].

Similar to our results on variable coefficient second order linear equations, operator factorization which reduces to solving the Riccati equation was outlined (Robin, 2007) [9]. Solution algorithm by quadratures of certain variable coefficient second order equations were depicted (Ward, 1984) [11]. Second order linear equations with coefficients being polynomials were also investigated in detail (Wilmer III and Costa, 2008) [12].

Theory of the Core-Shell Approach

It may be good to divide the embedded equations into two main categories first: a) Linear Embedded, b) Nonlinear Embedded. In this study, the linear embedded equations will be extensively analysed. The nonlinear case is left to further more advanced studies as in a typical undergraduate course, usually the linear equations are mainly treated with a possible exception of the first order equations. The application of the method to nonlinear equations is not straightforward and may not apply to all nonlinear equations. On the contrary, the shell-core approach can be applied theoretically to any linear differential equation of arbitrary order (excluding the first order equations) such as, constant coefficient equations, variable coefficient equations, homogenous equations, non-homogenous equations, etc.

Linear Embedded Equations

Consider the linear differential equation with two linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$

$$\mathcal{L}_2\mathcal{L}_1 y_1(x) = y_3(x) \text{ (Embedded equation)} \tag{2.1}$$

Where

$$\mathcal{L}_1 y_1(x) = y_2(x) \text{ (Core equation)} \tag{2.2}$$

is the core equation, and

$$\mathcal{L}_2 y_2(x) = y_3(x) \text{ (Shell equation)} \tag{2.3}$$

is the shell equation. Note that $y_3(x)$ is a given known function of the original equation. Once the shell equation is solved for $y_2(x)$ and substituted into the right hand side of the core equation, $y_1(x)$, which is the solution of the embedded equation is found. One may define the functions and operators:

$y_1(x)$: Solution of both the core and embedded equation

$y_2(x)$: Non-homogenous part of the core equation and solution of the shell equation

$y_3(x)$: Non-homogenous part of both the shell equation and the embedded equation

$\mathcal{L}_1$: The core differential operator

$\mathcal{L}_2$: The shell differential operator

There might be more than one shell operator. In that case, outer shell and intermediate shell terms may be used. Consider the linear embedded differential equation

$$\mathcal{L}_p\mathcal{L}_{p-1} \dots \mathcal{L}_2\mathcal{L}_1 y_1(x) = y_{p+1}(x) \text{ (Embedded equation)} \tag{2.4}$$

where $\mathcal{L}_p$ is the outer shell, $\mathcal{L}_{p-1} \dots \mathcal{L}_2$ are the intermediate shells and $\mathcal{L}_1$ is the core operator. The equations are decomposed into the following components

$$\mathcal{L}_1 y_1(x) = y_2(x) \text{ (Core equation)} \tag{2.5}$$

$$\mathcal{L}_2 y_2(x) = y_3(x) \text{ (Intermediate First Shell equation)} \tag{2.6}$$

$$\mathcal{L}_3 y_3(x) = y_4(x) \text{ (Intermediate Second Shell equation)} \tag{2.7}$$

$$\mathcal{L}_p y_p(x) = y_{p+1}(x) \text{ (Shell equation)} \tag{2.8}$$

Starting from the last shell equation, solving the equations one by one moving backwards, finally the core is reached whose solution $y_1(x)$ is the solution of the original embedded equation.

2.2. Nonlinear Embedded Equations

Consider the nonlinear embedded equation

$$F_1\left(x, y_1(x), y_1'(x) \dots y_1^{(k_1)}(x)\right) = y_3(x) \text{ (Embedded equation)} \tag{2.9}$$

Which, if possible, may be decomposed into the core and shell parts

$$F_2\left(x, y_1(x), y_1'(x) \dots y_1^{(k_2)}(x)\right) = y_2(x) \text{ (Core equation)} \tag{2.10}$$

$$F_3\left(x, y_2(x), y_2'(x) \ldots y_2^{(k_3)}(x)\right) = y_3(x) \text{ (Shell equation)} \tag{2.11}$$

with $k_1 = k_2 + k_3$. If one of the F_2 or F_3 are linear in their arguments, then one may speak of mixed embedded equations which may still be investigated under the nonlinear embedded equation title, since the original equation is nonlinear.
Since the nonlinear case will not be investigated in the subsequent sections, two simple examples will be given to clarify the issues.

Example 2.1
Consider the nonlinear equation with initial conditions:

$$y'' + y'^2 = 0 \; y(0) = 0, \; y'(0) = 1 \tag{2.12}$$

This is a mixed embedded equation with the core equation being linear

$$y_1' = y_2 \text{ (Core equation)} \tag{2.13}$$

and the shell equation being nonlinear

$$y_2' + y_2^2 = 0 \text{ (Shell equation)} \tag{2.14}$$

The shell equation is solved first

$$y_2 = \frac{1}{x + c_1} \tag{2.15}$$

and substituted into the core equation for which the solution is

$$y_1 = \ln(x + c_1) + c_2 \tag{2.16}$$

Applying the initial conditions, the final solution satisfying the equation and conditions is

$$y_1 = \ln(1 + x) \tag{2.17}$$

As seen from this basic example, the well-known reduction of order method can be re-interpreted within the context of the core-shell approach.

Example 2.2
For the nonlinear embedded equation

$$(1 + y'^2)y''' - 3y'y''^2 = 0 \tag{2.18}$$

if one can realize its core and shell parts

$$\frac{y_1''}{(1 + y_1'^2)^{3/2}} = y_2 \text{ (Core equation)} \tag{2.19}$$

$$y_2' = 0 \text{ (Shell equation)} \tag{2.20}$$

the solution can be found by solving the shell equation and substituting to the core equation. Indeed, the core equation may also be divided into shell and core parts by applying the reduction of order method.

General Theory for Second Order Linear Equations
First the solution formula for a second order variable coefficient differential equation will be given for the homogenous and non-homogenous cases and then the applications for the constant coefficient and variable coefficient cases are treated in detail.

***Theorem* 3.1**
For the variable coefficient homogenous second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3.1}$$

if the differential operator can be expressed in the core-shell form

$$\big(D + a(x)\big)\big(D + b(x)\big)y = 0 \tag{3.2}$$

where $D=d/dx$, then the general solution is of the below form

$$y(x) = c_1 e^{-\int b(x)dx} \int e^{\int (b(x)-a(x))dx} dx + c_2 e^{-\int b(x)dx} \tag{3.3}$$

The functions in the solutions satisfy the relations

$$p(x) = a(x) + b(x),$$

$$q(x) = b'(x) + a(x)b(x) \downarrow \tag{3.4}$$

Proof

The proof will be given by employing the core-shell approach. Note the relevant parts of the equation

$$\big(D + a(x)\big)\big(D + b(x)\big)y_1 = 0 \text{ (Embedded Equation)} \tag{3.5}$$

$$\big(D + b(x)\big)y_1 = y_2 \text{ (Core Equation)} \tag{3.6}$$

$$\big(D + a(x)\big)y_2 = 0 \text{ (Shell Equation)} \tag{3.7}$$

First the shell equation is solved

$$y_2' + a(x)y_2 = 0 \Rightarrow y_2 = c_1 e^{-\int a(x)dx} \tag{3.8}$$

and then the core equation

$$y_1' + b(x)y_1 = c_1 e^{-\int a(x)dx} \tag{3.9}$$

which is solved by multiplying the equation with the integrating factor $e^{\int b(x)dx}$

$y_1 = c_1 e^{-\int b(x)dx} \int e^{\int (b(x)-a(x))dx} dx + c_2 e^{-\int b(x)dx}$ (3.10) Furthermore

$$\big(D + a(x)\big)\big(D + b(x)\big)y = y'' + \big(a(x) + b(x)\big)y' + \Big(b'(x) + a(x)b(x)\Big)y = 0 \tag{3.11}$$

and comparing with the original equation $p = a + b$ and $q = b' + ab \downarrow$

Solution (3.3) was already reported previously for constant and variable second order equations (Lutzer, 2006; Figueroa &Rebolledo, 2015a; Robin, 2007, Ward, 1984) [5, 4, 9, 11]. For the special case of $a(x)=b(x)$, solution (3.3) reduces to

$$y(x) = c_1 x e^{-\int a(x)dx} + c_2 e^{-\int a(x)dx} \tag{3.12}$$

It should be noted that for constant operators, while the operators commute, $(D + a)(D + b) = (D + b)(D + a)$, this is not the case for variable operators

$$\big(D + a(x)\big)\big(D + b(x)\big) \neq (D + b(x))(D + a(x)).$$

For the non-homogenous form, the following theorem applies

Theorem 3.2

For the variable coefficient non-homogenous second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{3.13}$$

if the differential operator can be expressed in the core-shell form

$$\big(D + a(x)\big)\big(D + b(x)\big)y = f(x) \tag{3.14}$$

then the general solution is

$$y(x) = c_1 y_{1h}(x) + c_2 y_{2h}(x) + y_p(x) \tag{3.15}$$

Where

$$y_{1h}(x) = e^{-\int b(x)dx} \int e^{\int (b(x)-a(x))dx} dx \tag{3.16}$$

$$y_{2h}(x) = e^{-\int b(x)dx} \quad (3.17)$$

$$y_p(x) = e^{-\int b(x)dx} \int e^{\int (b(x)-a(x))dx} \int f(x) e^{\int a(x)dx}\, dx \quad (3.18)$$

With

$$p(x) = a(x) + b(x), q(x) = b'(x) + a(x)b(x) \downarrow \quad (3.19)$$

Proof

The proof for the homogenous solutions were already given previously. For the particular solution, the core and shell equations are

$$\big(D + b(x)\big) y_p = u(x) \text{ (core equation)} \quad (3.20)$$

$$\big(D + a(x)\big) u = f(x) \text{ (shell equation)} \quad (3.21)$$

The solution of the shell equation is

$$u = e^{-\int a(x)dx} \int f(x) e^{\int a(x)dx}\, dx \quad (3.22)$$

which upon substitution into the core equation

$$\big(D + b(x)\big) y_p = e^{-\int a(x)dx} \int f(x) e^{\int a(x)dx}\, dx \quad (3.23)$$

produces the solution

$$y_p(x) = e^{-\int b(x)dx} \int (e^{\int (b(x)-a(x))dx} \int f(x) e^{\int a(x)dx}\, dx) dx \downarrow \quad (3.24)$$

Similar results were reported previously (Lutzer, 2006; Figueroa & Rebolledo, 2015a; Robin, 2007, Ward, 1984) [5, 4, 9, 11]. This particular solution is identical to the solution obtained by the variation of parameters method. For the variation of parameters, the particular solution is

$$y_p = -y_{1h} \int \frac{y_{2h} f}{W} dx + y_{2h} \int \frac{y_{1h} f}{W} dx \quad (3.25)$$

where W is the Wronskian determinant defined by

$$W = y_{1h} y_{2h}' - y_{2h} y_{1h}' \quad (3.26)$$

Substituting (3.16) and (3.17), making the simplifications, the Wronskian turns out to be

$$W(x) = e^{-\int (a(x)+b(x))dx} \quad (3.27)$$

Finally, inserting (3.16), (3.17) and (3.27) into (3.25) leads to the particular solution identical with (3.24).

Applications for Second Order Linear Equations

Applications of the general theorems as well as the new way of approach are depicted for linear second order differential equations in this section.

Second Order Constant Coefficient Equations

If $a(x)=a_0$ and $b(x)=b_0$, a_0 and b_0 being constants, the differential equation is

$(D + a_0)(D + b_0)y = y'' + (a_0 + b_0)y' + a_0 b_0 y = 0$ (4.1) whose solution turns out to be from Theorem 3.1

$$y(x) = c_1 e^{-\int b_0 dx} \int e^{\int (b_0 - a_0)dx} dx + c_2 e^{-\int b_0 dx} \quad (4.2)$$

Performing the integrals and re-defining c_1, the solution is

$$y(x) = c_1 e^{-a_0 x} + c_2 e^{-b_0 x} \quad (4.3)$$

which is the well- known solution for homogenous linear equations possessing roots of
$-a_0$ and $-b_0$ for the characteristic equation with $a_0 \neq b_0$. Although (4.3) is not valid for

$a_0 = b_0$, Eq. (4.2) still holds for this case also. However, in the decomposition of the second order operator, $a(x)$ and $b(x)$ need not be constants and they may be variables, yet the outcoming equation is still a constant coefficient equation. From Theorem 3.1, one recalls that $a(x)$ and $b(x)$ should satisfy

$$p_0 = a(x) + b(x),\ q_0 = b'(x) + a(x)b(x) \tag{4.4}$$

with p_0 and q_0 are constants for the constant coefficient linear equation

$$y'' + p_0 y' + q_0 y = 0 \tag{4.5}$$

Examples will be given to clarify the issue

Example 4.1.1
Consider the equation

$$y'' + y = 0 \tag{4.6}$$

If one looks for a constant coefficient decomposition, then from (4.4)

$$0 = a_0 + b_0\ 1 = a_0 b_0 \tag{4.7}$$

which dictates a solution $a_0 = i$ and $b_0 = -i$. In the operator notation, the equation is

$$(D + i)(D - i)y = 0 \tag{4.8}$$

with a solution from (4.3)

$$y(x) = c_1 e^{-ix} + c_2 e^{ix} \tag{4.9}$$

This solution reduces to the harmonic solution by employing the Euler formula and redefining the constants

$$y(x) = c_1 cosx + c_2 sinx. \tag{4.10}$$

If one does not want to deal with complex quantities, then a real decomposition, albeit variable, is already available. If $a(x) = -tanx$, $b(x) = tanx$, equation (4.4) is again satisfied. Hence for

$$(D - tanx)(D + tanx)y = (D^2 + 1)y = y'' + y = 0 \tag{4.11}$$

the solution (3.3) reads

$$y(x) = c_1 e^{-\int tanxdx} \int e^{\int 2tanxdx} dx + c_2 e^{-\int tanxdx} \tag{4.12}$$

Performing the integrals

$$y(x) = c_1 cosx \int \frac{1}{cos^2 x} dx + c_2 cosx \tag{4.13}$$

The final result is

$$y(x) = c_1 sinx + c_2 cosx \tag{4.14}$$

which is the same result obtained by complex constant roots ↓
Inspired from the example, the following theorem can be stated:

Theorem 4.1
The second order constant coefficient linear differential equation

$$y'' + p_0 y' + q_0 y = 0 \tag{4.15}$$

can be expressed in operator form

$$(D + a)(D + b)y = 0 \tag{4.16}$$

where *a* and *b* can be either
i) constants (may be real or imaginary numbers)

$$a = \frac{1}{2}\left(p_0 + \sqrt{p_0^2 - 4q_0}\right),\ b = \frac{1}{2}\left(p_0 - \sqrt{p_0^2 - 4q_0}\right) \tag{4.17}$$

ii) variables

$$b = b_0 + \frac{2b_0 - p_0}{c(2b_0 - p_0)e^{-(2b_0 - p_0)x} - 1},\ a = p_0 - b \tag{4.18}$$

where

$$b_0 = \frac{1}{2}\left(p_0 \mp \sqrt{p_0^2 - 4q_0}\right) \downarrow \tag{4.19}$$

Proof
The first part is straightforward and left as an exercise. For the second part, from (4.4)

$$p_0 = a(x) + b(x),\ q_0 = b'(x) + a(x)b(x) \tag{4.20}$$

Substituting $a = p_0 - b$ to the second equation leads to a Riccati equation

$$b' + p_0 b = q_0 + b^2. \tag{4.21}$$

One constant solution is

$$b_0 = \frac{1}{2}\left(p_0 \mp \sqrt{p_0^2 - 4q_0}\right) \tag{4.22}$$

and the other solution can be obtained by the transformation

$$b = b_0 + \frac{1}{v} \tag{4.23}$$

which leads to

$$v' + (2b_0 - p_0)v = -1 \tag{4.24}$$

Solving and substituting into (4.23) finally leads to (4.18) ↓

In fact, finding the decomposed operators reduces to a problem of solving a Riccati equation. The importance of the Riccati equation stems from the fact that the solution of any second order variable coefficient linear equation is associated with the solution of the Riccati equation. From the educational point of view, this property of the Riccati equation should be emphasized in the classrooms. For a detailed analysis on Riccati equations and possible solutions, see Ndiaye (2022) [7]. The constant *c* in (4.18) is arbitrary and can be assigned a special value to simplify the calculations and the form of the operators.

Example 4.1.2
For the differential equation

$$y'' + 3y' + 2y = 0 \tag{4.25}$$

with p_0=3 and q_0=2, b_0=1 or 2 from (4.19). Take b_0=1, then

$$b = 1 + \frac{1}{ce^x + 1},\ a = 2 - \frac{1}{ce^x + 1} \tag{4.26}$$

from (4.18) ↓

Example 4.1.3
For the differential equation

$$y'' + 2y' + 2y = 0 \tag{4.27}$$

with p_0=2 and q_0=2, b_0=1+i. Then

$$b = \frac{4c^2 + 4c(-sin2x + cos2x) + i(4c^2 - 1)}{4c^2 - 4csin2x + 1}, \tag{4.28}$$

from (4.18). One may choose now c=1/2 so that the decomposition is real

$$b = \frac{1+2(-sin2x+cos2x)}{2(1-sin2x)}, a = \frac{3-2(sin2x+cos2x)}{2(1-sin2x)} \downarrow \quad (4.29)$$

The following theorem ensures a real operator expression for a constant coefficient second order linear equation.

Theorem 4.2
For the second order constant coefficient linear differential equation

$$y'' + p_0 y' + q_0 y = 0 \quad (4.30)$$

whose characteristic equation possesses imaginary roots, i.e.

$$p_0^2 - 4q_0 < 0 \quad (4.31)$$

the second order differential operator $D^2 + p_0 D + q_0$ can always be decomposed into first order operators

$$(D + a(x))(D + b(x))y = 0 \quad (4.32)$$

where *a*(*x*) and *b*(*x*) are real variable quantities and defined as

$$b = \frac{p_0}{2} + \frac{\sqrt{4q_0 - p_0^2}cos\left(\sqrt{4q_0 - p_0^2}\,x\right)}{2\left(1-sin\left(\sqrt{4q_0-p_0^2}\,x\right)\right)} \text{ or } b = \frac{p_0}{2} - \frac{\sqrt{4q_0 - p_0^2}cos\left(\sqrt{4q_0 - p_0^2}\,x\right)}{2\left(1+sin\left(\sqrt{4q_0-p_0^2}\,x\right)\right)} \quad (4.33)$$

$$a = p_0 - b \quad (4.34)$$

Proof
Select $b_0 = \frac{1}{2}\left(p_0 + \sqrt{p_0^2 - 4q_0}\right)$ without loss of generality from (4.19). From (4.18)

$$b = b_0 + \frac{2b_0 - p_0}{c(2b_0-p_0)e^{-(2b_0-p_0)x} - 1} \quad (4.35)$$

$2b_0 - p_0 = \sqrt{p_0^2 - 4q_0} = ki$ since $p_0^2 - 4q_0 < 0$ with $k^2 = 4q_0 - p_0^2$. Substituting all into (4.35), employing the Euler formula, making the denominator real by multiplying with its complex conjugate and rearranging

$$b = \frac{\frac{1}{2}p_0(c^2k^2+1-2cksinkx)+\frac{1}{2}ki(c^2k^2-1)+ck^2coskx}{c^2k^2+1-2cksinkx}. \quad (4.36)$$

To eliminate the imaginary part, one may choose the arbitrary coefficient

$$c = \mp\frac{1}{k} \quad (4.37)$$

eventually leading to (4.33) and (4.34) ↓

Second Order Variable Coefficient Equations
Some sample problems with variable coefficients will be treated in this sub-section.

Example 4.2.1 Cauchy-Euler Differential Equation
Consider the Cauchy-Euler differential equation

$$y'' + \frac{p_0}{x}y' + \frac{q_0}{x^2}y = 0 \quad (4.38)$$

for which the standard solution is achieved by a transformation *t=lnx* (ONeil, 1991) [8]. For the application of the core-shell approach, from Theorem 3.1, first the *a*(*x*) and *b*(*x*) should be determined

$$a + b = \frac{p_0}{x}, b' + ab = \frac{q_0}{x^2} \quad (4.39)$$

Substituting *a*(*x*) from the first equation into the second one

$$b' + \frac{p_0}{x}b = \frac{q_0}{x^2} + b^2 \quad (4.40)$$

yields the Riccati equation. Try solutions of the form *b=c/x* with the constant *c* satisfying

$$c^2 + (1 - p_0)c + q_0 = 0 \tag{4.41}$$

Solving c and substituting to b=c/x yields

$$b = \frac{1}{2x}\left(p_0 - 1 + \sqrt{(1-p_0)^2 - 4q_0}\right), a = \frac{1}{2x}\left(p_0 + 1 - \sqrt{(1-p_0)^2 - 4q_0}\right) \tag{4.42}$$

For these specific functions, the integrals in the solution, i.e. Eq. (3.3), are performed with ease finally yielding

$$y = c_1 x^{-\frac{1}{2}\left(p_0 - 1 - \sqrt{(1-p_0)^2 - 4q_0}\right)} + c_2 x^{-\frac{1}{2}\left(p_0 - 1 + \sqrt{(1-p_0)^2 - 4q_0}\right)} \tag{4.43}$$

which is the solution of the general Cauchy-Euler equation($(1 - p_0)^2 - 4q_0 \neq 0$). For the specific example of

$$y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0 \tag{4.44}$$

p_0=2, q_0=-6, and the solution is from (4.43)

$$y = c_1 x^2 + c_2 x^{-3} \downarrow \tag{4.45}$$

Example 4.2.2
Consider the differential equation

$$y'' + 2xy' + (x^2 + 1)y = 0 \tag{4.46}$$

For the application of the core-shell approach, the decomposed operator is

$$(D + x)(D + x)y = 0 \tag{4.47}$$

with

$$a = x, b = x \tag{4.48}$$

Equation (3.3) yields

$$y = c_1 x e^{-x^2/2} + c_2 e^{-x^2/2} \downarrow \tag{4.49}$$

In the previous two examples outlined above, the equation is given and the solution is found from the formula of Theorem 3.1. Indeed, variable coefficient second order equations for which explicit analytical solutions exist are rare and they correspond to $a(x)$ and $b(x)$ functions that can be integrated analytically in equation (3.3). For instructors, when designing new problems which possesses an explicit analytical solution, the algorithm will be

i) Select suitable $a(x)$ and $b(x)$ so that (3.3) is integrable.
ii) Express the equation in the operator notation $(D + a(x))(D + b(x))y = 0$.
iii) Finally write the solution employing (3.3).

Example 4.2.3
The following equation which has an analytical solution is extracted starting by selecting

$$a = x, b = \frac{1}{x} \tag{4.50}$$

which are functions that can be easily integrable. The corresponding equation is

$$(D + x)\left(D + \frac{1}{x}\right)y = y'' + \left(\frac{1}{x} + x\right)y' + \left(1 - \frac{1}{x^2}\right)y = 0 \tag{4.51}$$

and the solution is retrieved from (3.3)

$$y = \frac{1}{x}\left(c_1 e^{-x^2/2} + c_2\right) \downarrow \tag{4.52}$$

***Example* 4.2.4**
If $a = cosx, b = cosx$ (4.53)
which can be easily integrable, the corresponding equation is

$$(D + cosx)(D + cosx)y = y'' + 2cosxy' + (cos^2x - sinx)y = 0 \tag{4.54}$$

and the solution is

$$y = e^{-sinx}(c_1x + c_2) \downarrow \quad (4.55)$$

Functions $a(x)$ and $b(x)$ may be imaginary also. Note the following example

Example 4.2.5

Consider the differential equation

$$y'' - \frac{1}{x}y' + 4x^2y = 0 \quad (4.56)$$

for which

$$a + b = -\frac{1}{x}, b' + ab = 4x^2 \quad (4.57)$$

Eliminating $a(x)$ between the equations

$$b' - \frac{1}{x}b - b^2 = 4x^2 \quad (4.58)$$

for which the solutions are complex

$$a = -\frac{1}{x} - 2ix, b = 2ix \quad (4.59)$$

The decomposed form is

$$\left(D - \frac{1}{x} - 2ix\right)(D + 2ix)y = 0 \quad (4.60)$$

with the solution retrieved from (3.3) by redefining the constants

$$y = c_1cosx^2 + c_2sinx^2 \downarrow \quad (4.61)$$

Third Order Linear Equations

Third order equations with variable coefficients can be solved with a similar approach as presented in the previous sections. For the third order equations, the following theory can be proposed

***Theorem* 5.1**

For the variable coefficient homogenous third order linear differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0 \quad (5.1)$$

if the differential operator can be expressed in the core-shell form

$$\big(D + a(x)\big)\big(D + b(x)\big)\big(D + c(x)\big)y = 0 \quad (5.2)$$

where $D=d/dx$, then the general solution is of the below form

$$y(x) = c_1e^{-\int c(x)dx}\int(e^{\int(c(x)-b(x))dx}\int e^{\int(b(x)-a(x))dx}dx)dx + c_2e^{-\int c(x)dx}\int e^{\int(c(x)-b(x))dx}dx + c_3e^{-\int c(x)dx} \quad (5.3)$$

where

$$p(x) = a + b + c \quad (5.4)$$

$$q(x) = 2c' + b' + ab + c(a + b) \quad (5.5)$$

$$r(x) = c'' + (a + b)c' + (b' + ab)c \downarrow \quad (5.6)$$

Proof

The embedded equation

$$\big(D + a(x)\big)\big(D + b(x)\big)\big(D + c(x)\big)y_1 = 0 \quad (5.7)$$

can be divided into core and shell parts

$$\big(D + a(x)\big)\big(D + b(x)\big)y_2 = 0 \text{ (Shell Equation)} \quad (5.8)$$

$$\big(D + c(x)\big)y_1 = y_2 \text{ (Core Equation)} \tag{5.9}$$

The solution of the shell equation was already given in Theorem 3.1

$$y_2 = c_1 e^{-\int b(x)dx} \int e^{\int (b(x)-a(x))dx} dx + c_2 e^{-\int b(x)dx} \tag{5.10}$$

Hence, one needs to solve the core equation

$$\big(D + c(x)\big)y_1 = c_1 e^{-\int b(x)dx} \int e^{\int (b(x)-a(x))dx} dx + c_2 e^{-\int b(x)dx} \tag{5.11}$$

The equation is a linear non-homogenous first order equation which can be solved by the standard integrating factor method yielding (5.3). The straightforward calculation of the operator form (5.2) and comparison with the original equation (5.1) yields the definitions given in (5.4)-(5.6) ↓

The idea can easily be generalized to arbitrary order of linear equations with variable coefficients. However, the functional relationships of the decomposed operators, i.e. Eqs (5.4)-(5.6) would become more and more complex as the order increases.

***Example* 5.1**

Consider the differential equation

$$y''' + 3xy'' + 3(x^2+1)y' + x(x^2+3)y = 0 \tag{5.12}$$

for which

$$3x = a + b + c \tag{5.13}$$

$$3(x^2+1) = 2c' + b' + ab + c(a+b) \tag{5.14}$$

$$x(x^2+3) = c'' + (a+b)c' + (b' + ab)c \tag{5.15}$$

The solution is

$$a = x, b = x, c = x \tag{5.16}$$

The decomposed form is

$$(D + x)(D + x)(D + x)y = 0 \tag{5.17}$$

with the solution an immediate consequence of (5.3)

$$y = e^{-x^2/2}(c_1x^2 + c_2x + c_3) \downarrow \tag{5.18}$$

***Example* 5.2**

Consider the differential equation

$$y''' + 3sinxy'' + 3(sin^2x + cosx)y' + (3sinxcosx + sin^3x - sinx)y = 0 \text{ (5.19)}$$

The functions of the decomposed operators satisfy

$$3sinx = a + b + c \tag{5.20}$$

$$3(sin^2x + cosx) = 2c' + b' + ab + c(a+b) \tag{5.21}$$

$$3sinxcosx + sin^3x - sinx = c'' + (a+b)c' + (b' + ab)c \quad \text{(5.22)}$$

The solutions are

$$a = sinx, b = sinx, c = sinx \tag{5.23}$$

The decomposed form is

$$(D + sinx)(D + sinx)\ (D + sinx)\ y = 0 \tag{5.24}$$

with the final solution from (5.3)

$$y = e^{cosx}(c_1x^2 + c_2x + c_3) \downarrow \tag{5.25}$$

Note that, for third order equations, solutions of (5.4)-(5.6) to determine a, b and c might not be trivial for most of the cases. If found, there is no guarantee that the integrals can be evaluated in closed form solutions in (5.3). This explains the rarity of the analytical solutions for variable coefficient equations.

Suggestions for Further Research

Some research topics regarding the core-shell approach may be proposed:

- The method of undetermined coefficients for finding particular solutions of the linear differential equations can be re-interpreted within the context of core-shell approach.
- The perturbation solutions can be re-interpreted within the core-shell approach.
- The inner and outer expansions in a boundary layer type equation (singular perturbation problems) can be re-interpreted within the core-shell approach.
- The calculation of symmetries of the differential equations and constructing the exact solutions from the symmetries can be re-interpreted within the core-shell approach.
- New numerical algorithms based on the formulations of the core-shell approach may be developed.

Except the first one, which is a topic of the fundamental course on differential equations, others may be given to advanced senior graduate students as research projects.

Concluding Remarks

A new approach to the teaching of differential equations are given which is named as the core-shell approach. Most of the well-established techniques in search of analytical solutions can be re-interpreted within the context of the new approach. This might improve the understanding of the differential equations and their solutions. The formalism given in the text for linear differential equations may be useful for instructors to design new differential equations possessing exact solutions. It is shown that if a differential equation possesses an exact solution, then some specific integrals associated with the functions appearing in the operators must be integrable. Finally, some further topics are suggested for instructors as research projects to be assigned to the undergraduate students.

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