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Multi-timing perturbation and asymptotic analysis of the dynamic buckling of an elastic cubic model structure modulated by a dynamically slowly-varying oscillatory load

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Abstract
This investigation examines the dynamic buckling of an imperfect cubic model elastic structure trapped by a time-dependent but slowly varying oscillatory load applied just after the initial time. Besides its oscillatory nature, the amplitude of the load is assumed to be also strictly slowly varying dynamically and has right hand derivatives of all orders evaluated at the initial time. A multi-timing perturbation approach is adopted in asymptotic expansions of the variables. All results are thus asymptotic in nature. In the final analysis, a simple result for determining the dynamic buckling is obtained.

Keywords: Dynamic buckling load, perturbation, asymptotic analysis, slowly varying oscillatory load

Introduction
Investigations into dynamic buckling of structures have received accelerated progress in the past fifty years, yet so much seems yet to be investigated or explored. In this analysis, we beam our search light on a loading history that is dynamically slowly varying and oscillatory in nature. The load amplitude which is also slowly varying is explicitly time-dependent, continuous, monotone-decreasing and infinitely differentiable at the initial time.

To our knowledge, Kuzmak [1] first propounded the general asymptotic theory for solving nonlinear second order differential equations with variable coefficients.

Later works include investigations by Luke [2], Kevorkian [3], Kevorkian and Li [4], Boslly [5] and Kroll *et al* [6], among others.

We recall that Ozoigbo *et al*. [7] analysed a pre-statically loaded nonlinear cubic structure pressurized by an explicitly time dependent slowly varying load while Ozoigbo and Ette [8] investigated the perturbation approach to dynamic buckling of a statically pre-loaded but viscously damped elastic structure. Similar investigations are those of Ette *et al* [9-12] and Amazigo and Ette [13], where similar perturbation and asymptotic analysis were adopted. The numerical approach adopted by Kolakowski [14], Simitses [16-17], Tabiei *et al*. [18], Groh and Croll [18] among others, are similarly insightful and could provide a comparative analysis.

Formulation of the Problem

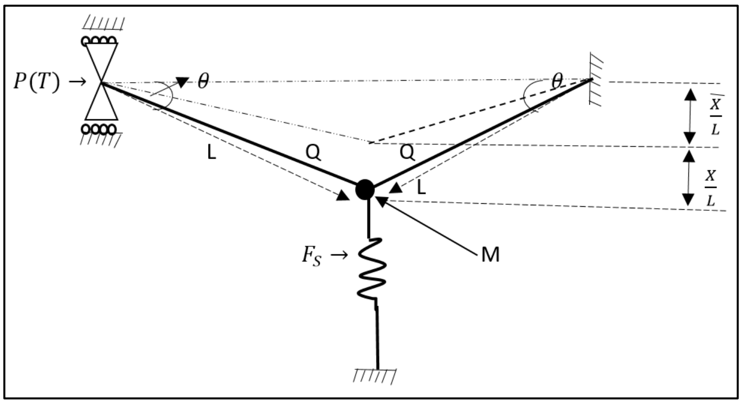

Fig 1: Simple Cubic Model Structure

The elastic cubic model structure (Fig. 1) considered in this work was first studied by Budiansky [19] and Hutchinson and Budiansky [20] and has continued to serve as a generalized mathematical model of most physical elastic structures encountered in engineering practices.

The system consists of a two-arm simple mechanical arrangement, which has two elastic rods, each arm of length L, arranged as in the figure and carrying a mass M at their meeting point while the ensuring vertical motion is restrained by a string whose rigidity follows a cubic law.

On the spring, the mass is affected by the reaction force given by

$$F_S = KL\left(\frac{X}{L} - b\left(\frac{X}{L}\right)^3\right), K > 0, \beta > 0$$

where, K is the spring constant and $\frac{X}{L}$ is the additional displacement from equilibrium position. As in the Fig.1, $\frac{\bar{X}}{L}$ is the initial displacement serving as the initial imperfection. The angle θ is assumed very small such that $\cos\theta \approx 1, \sin\theta \approx \theta$. Letting Q be the tension on each arm of the rod, we have $Q\cos\theta = P(T)$. Consequently, the equation of motion is

$$M\frac{d^2}{dT^2}\left(\frac{X}{L}\right) + KL\left(1 - \frac{2P(T)}{KL^2}\right)\left(\frac{X}{L}\right) - bKL\left(\frac{X}{L}\right)^3 = 2P(T)\left(\frac{\bar{X}}{L}\right)$$

The following are the nondimensional quantities:

$$\xi = \frac{X}{L}, \bar{\xi} = \frac{\bar{X}}{L}, \hat{t} = T\sqrt{\frac{KL}{M}}, \lambda = \frac{2P(0)}{KL}, \frac{P(T)}{P(0)} = f(\delta\hat{t})\cos\delta\hat{t},$$
$$0 < \lambda < 1, 0 < \bar{\xi} < 1, P(0) \neq 0,$$

Thus, the governing nondimensional equation of motion follows in the form

$$\frac{d^2\xi}{d\hat{t}^2} + (1 - \lambda f(\delta\hat{t})\cos\delta\hat{t})\xi - b\xi^3 = \lambda\bar{\xi}f(\delta\hat{t})\cos\delta\hat{t}, \hat{t} > 0 \quad (1)$$

$$\xi(0) = \frac{d\xi(0)}{d\hat{t}} = 0. \quad (2)$$

Here, b>0 serves as the imperfection sensitivity parameter, λ is the amplitude of the load $f(\delta\hat{t})\cos\delta\hat{t}$ while $\xi(\hat{t})$ is the displacement as a function of time $\hat{t}$ and our aim is to determine a certain value of the load parameter λ, called the dynamic buckling load upon which the structure buckles dynamically. As in Ozoigbo [7], the dynamic buckling load is defined as the highest value of load parameter for the displacement $\xi(\hat{t})$ to be bounded and is obtained from the condition

$$\frac{d\lambda}{d\xi_a} = 0 \quad (3)$$

where ξ_a is the maximum displacement as a function of time.

The problem (1) and (2) is nonlinear with dynamically slowly varying oscillatory coefficients which will now be solved using multi-timing regular perturbations in asymptotic expansions involving the small parameters $\bar{\xi}$ and δ, where $\bar{\xi}$ is the nondimensional initial displacement.

Perturbations and Asymptotic Expansions

For the sake of clarity, we again rewrite (1) and (2) as

$$\frac{d^2\xi}{d\hat{t}^2} + (1 - \lambda f(\delta\hat{t})\cos\delta\hat{t})\xi - b\xi^3 = \lambda\bar{\xi}f(\delta\hat{t})\cos\delta\hat{t}, \hat{t} > 0, b > 0$$
$$\xi(0) = \frac{d\xi(0)}{d\hat{t}} = 0, 0 < \delta < 1, 0 < \bar{\xi} < 1,$$
$$f(0) = 1, |f(\delta\hat{t})| < 1, \hat{t} > 0$$

Let

$$\tau = \delta\hat{t}, \quad (4)$$

$$\frac{d\bar{t}}{d\hat{t}} = (1 - \lambda f(\delta\hat{t})\cos\delta\hat{t})^{\frac{1}{2}} = (1 - \lambda f(\tau)\cos\tau)^{\frac{1}{2}} \quad (5)$$

Further, let

$$t = \bar{t} + \frac{1}{\delta}\left(\bar{\xi}^2\mu_2(\tau) + \bar{\xi}^3\mu_3(\tau) + \cdots\right) \quad (6)$$

$$\mu_i(0) = 0, i = 2, 3, \dots \tag{7}$$

Now let

$$\xi(\hat{t}) = \eta(t,\tau) \tag{8}$$

Hence,

$$\frac{d\xi}{d\hat{t}} = \frac{\partial\eta}{\partial t}\frac{\partial t}{\partial \bar{t}}\frac{d\bar{t}}{d\hat{t}} + \frac{\partial\eta}{\partial t}\frac{\partial t}{\partial \tau}\frac{d\tau}{d\hat{t}} + \frac{\partial\eta}{\partial \tau}\frac{d\tau}{d\hat{t}}$$
$$=(1-\lambda f(\tau)\cos\tau)^{\frac{1}{2}}\eta_t + \left(\mu'_2\bar{\xi}^2 + \mu'_3\bar{\xi}^3 + \cdots\right)\eta_t + \delta\eta_\tau \tag{9}$$

where, $\eta_t = \frac{\partial\eta}{\partial t}$, $\mu'_i = \frac{d\mu_i}{d\tau}$, $\eta_\tau = \frac{\partial\eta}{\partial\tau}$, $i = 2, 3, \dots$.

Thus,

$$\frac{d^2\xi}{d\hat{t}^2} = (1-\lambda f\cos\tau)\eta_{tt} + \left(\mu'_2\bar{\xi}^2 + \mu'_3\bar{\xi}^3 + \cdots\right)^2\eta_{tt} + \delta^2\eta_{\tau\tau} + 2\delta\left(\mu'_2\bar{\xi}^2 + \mu'_3\bar{\xi}^3 + \cdots\right)\eta_{t\tau}$$
$$+2(1-\lambda f\cos\tau)^{\frac{1}{2}}\left(\mu'_2\bar{\xi}^2 + \mu'_3\bar{\xi}^3 + \cdots\right)\eta_{tt} + 2\delta(1-\lambda f\cos\tau)^{\frac{1}{2}}\eta_{t\tau}$$
$$+\delta\left(\mu''_2\bar{\xi}^2 + \mu''_3\bar{\xi}^3 + \cdots\right)\eta_t - \frac{\delta\lambda(f'\cos\tau - f\sin\tau)}{2(1-\lambda f\cos\tau)^{\frac{1}{2}}}\eta_t \tag{10}$$

Therefore, substituting (9) and (10) into (1) and (2) and simplifying yields

$$\eta_{tt} + \frac{\left(\mu'_2\bar{\xi}^2 + \mu'_3\bar{\xi}^3 + \cdots\right)^2\eta_{tt}}{F} + \frac{\delta^2\eta_{\tau\tau}}{F} + \frac{2\left(\mu'_2\bar{\xi}^2 + \mu'_3\bar{\xi}^3 + \cdots\right)\eta_{tt}}{F^{\frac{1}{2}}} + \frac{2\delta\left(\mu'_2\bar{\xi}^2 + \mu'_3\bar{\xi}^3 + \cdots\right)\eta_{t\tau}}{F}$$
$$+\frac{2\delta\eta_{t\tau}}{F^{\frac{1}{2}}} - \frac{\delta\lambda(f'\cos\tau - f\sin\tau)}{F^{\frac{3}{2}}}\eta_t + \frac{\delta\left(\mu''_2\bar{\xi}^2 + \mu''_3\bar{\xi}^3 + \cdots\right)\eta_t}{F} + \eta - \frac{b\eta^3}{F} = \frac{\lambda\bar{\xi}f\cos\tau}{F} \tag{11}$$

where,

$$F = (1-\lambda f\cos\tau),\quad F(0) = (1-\lambda) \tag{12}$$

Thus, the original problem in $\hat{t}$ now becomes a two-timing problem in t and τ.
Let,

$$\xi(\hat{t}) \equiv \eta(t,\tau) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\zeta^{(i,j)}(t,\tau)\bar{\xi}^i\delta^j \tag{13}$$

where the (i,j) in $\zeta^{(i,j)}$ are not powers but superscripts.
Substituting (13) in (11) and (12), and equating coefficients of $\bar{\xi}^i\delta^j$ we have:

$$O(\bar{\xi}):\ \zeta^{10}_{tt} + \zeta^{10} = B(\tau) = \frac{\lambda f\cos\tau}{F(\tau)} \tag{14}$$

$$O(\bar{\xi}\delta):\ \zeta^{11}_{tt} + \zeta^{11} = \frac{-2\zeta^{10}_{t\tau}}{F^{\frac{1}{2}}} + \frac{\lambda(f'\cos\tau - f\sin\tau)}{2F^{\frac{3}{2}}}\zeta^{10}_t \tag{15}$$

$$O(\bar{\xi}\delta^2):\ \zeta^{12}_{tt} + \zeta^{12} = \frac{-2\zeta^{11}_{t\tau}}{F^{\frac{1}{2}}} + \frac{\lambda(f'\cos\tau - f\sin\tau)}{2F^{\frac{3}{2}}}\zeta^{11}_t - \frac{\zeta^{10}_{\tau\tau}}{F} \tag{16}$$

$$O(\bar{\xi}^3):\ \zeta^{30}_{tt} + \zeta^{30} = \frac{-2\mu'_2\zeta^{10}_{tt}}{F^{\frac{1}{2}}} + \frac{b\left(\zeta^{10}\right)^3}{F} \tag{17}$$

$$O(\bar{\xi}^3\delta):\ \zeta^{31}_{tt} + \zeta^{31} = \frac{-2\mu'_2\zeta^{11}_{tt}}{F^{\frac{1}{2}}} + \frac{3b\left(\zeta^{10}\right)^2\zeta^{11}}{F} - \frac{2\mu'_2\zeta^{10}_{t\tau}}{F} - \frac{2\zeta^{30}_{t\tau}}{F^{\frac{1}{2}}} + \frac{\lambda(f'\cos\tau - f\sin\tau)\zeta^{30}_t}{2F^{\frac{3}{2}}} - \frac{\mu''_2\zeta^{10}_t}{F} \tag{18}$$

$$O(\bar{\xi}^3\delta^2):\ \zeta^{32}_{tt} + \zeta^{32} = \frac{\zeta^{30}_{\tau\tau}}{F} - \frac{2\mu'_2\zeta^{11}_{t\tau}}{F} - \frac{2\mu'_2\zeta^{12}_{tt}}{F^{\frac{1}{2}}} - \frac{2\zeta^{31}_{t\tau}}{F^{\frac{1}{2}}} + \frac{\lambda(f'\cos\tau - f\sin\tau)\zeta^{31}_t}{2F^{\frac{3}{2}}} - \frac{\mu''_2\zeta^{11}_t}{F}$$
$$+\frac{3b\left[\left(\zeta^{10}\right)^2\zeta^{12} + \left(\zeta^{11}\right)^2\zeta^{10}\right]}{F} \tag{19}$$

The initial conditions are:

$$\zeta^{(i,j)}(0,0) = 0, i = 1,2,3,\dots, j = 0,1,2,\dots \tag{20}$$

$$O(\bar{\xi}):\ \zeta^{10}_t = 0 \tag{21}$$

$$O(\bar{\xi}\delta): \zeta_t^{11}(0,0) + \frac{\zeta_\tau^{10}(0,0)}{(1-\lambda)^{\frac{1}{2}}} = 0, \tag{22}$$

$$O(\bar{\xi}\delta^2): \zeta_t^{12}(0,0) + \frac{\zeta_\tau^{11}(0,0)}{(1-\lambda)^{\frac{1}{2}}} = 0, \tag{23}$$

$$O(\bar{\xi}^3): \zeta_t^{30}(0,0) + \frac{\mu_2'(0)\zeta_t^{10}(0,0)}{(1-\lambda)^{\frac{1}{2}}} = 0, \tag{24}$$

$$O(\bar{\xi}^3\delta): \zeta_t^{31}(0,0) + \frac{\mu_2'(0)\zeta_t^{11}(0,0)+\zeta_\tau^{30}(0,0)}{(1-\lambda)^{\frac{1}{2}}} = 0, \tag{25}$$

$$O(\bar{\xi}^3\delta^2): \zeta_t^{32}(0,0) + \frac{\mu_2'(0)\zeta_t^{12}(0,0)+\zeta_\tau^{31}(0,0)}{(1-\lambda)^{\frac{1}{2}}} = 0, \tag{26}$$

etc.

Solution of the System of Perturbation Equations

The solution of (14) subject to (20) and (21) is:

$$\zeta^{10}(t,\tau) = \alpha_{10}(\tau)\cos t + \beta_{10}(\tau)\sin t + B(\tau) \tag{27a}$$

$$\alpha_{10}(0) = -B(0) = \frac{-\lambda}{1-\lambda}\,, \beta_{10}(0) = 0 \tag{27b}$$

Substituting into (15) and simplifying gives

$$\zeta_{tt}^{11} + \zeta^{11} = \frac{2\,(\alpha_{10}'\sin t - \beta_{10}'\cos t)}{F^{\frac{1}{2}}} + \frac{\lambda(f'\cos\tau - f\sin\tau)(-\alpha_{10}\sin t + \beta_{10}\cos t)}{F^{\frac{3}{2}}} \tag{28a}$$

To ensure a uniformly valid solution in t, we equate to zero, in (28a), the coefficients of $\cos t$ and $\sin t$ and respectively obtain:

$$\beta_{10}' - \frac{\lambda(f'\cos\tau - f\sin\tau)\beta_{10}}{4F} = 0 \tag{28b}$$

and

$$\alpha_{10}' - \frac{\lambda(f'\cos\tau - f\sin\tau)\alpha_{10}}{4F} = 0 \tag{28c}$$

The solutions to (28b, c) are

$$\beta_{10}(\tau) = 0, \alpha_{10}(\tau) = \frac{B(0)}{(1-\lambda f\cos\tau)^{\frac{1}{4}}} \tag{28d}$$

Thus, we conclude that

$$\zeta^{10}(t,\tau) = \alpha_{10}(\tau)\cos t + B(\tau) \tag{29}$$

Solving the remaining equation in (28a) gives

$$\zeta^{11}(t,\tau) = \alpha_{11}(\tau)\cos t + \beta_{11}(\tau)\sin t$$

$$\alpha_{11}(0) = 0\,, \beta_{11}(0) = -\frac{\alpha_{10}'(0)+B'(0)}{(1-\lambda)^{\frac{1}{2}}} = \frac{-B(0)f'(0)(4-\lambda)}{4\,(1-\lambda)^{\frac{3}{2}}} \tag{30a}$$

where

$$\alpha_{10}'(0) = \frac{-B^2(0)f'(0)}{4}, B'(0) = \frac{B(0)f'(0)}{(1-\lambda)} \tag{30b}$$

Next, substituting into (16) and simplifying gives

$$\zeta_{tt}^{12} + \zeta^{12} = \frac{-2\,(-\alpha_{11}'\sin t + \beta_{11}'\cos t)}{F^{\frac{1}{2}}} + \frac{\lambda(f'\cos\tau - f\sin\tau)(-\alpha_{11}\sin t + \beta_{11}\cos t)}{F^{\frac{3}{2}}} - \frac{(\alpha_{10}'\cos t + B'')}{F} \tag{31}$$

To ensure a uniformly valid solution in t, we equate to zero in (31) the coefficients of $\cos t$ and $\sin t$ and obtain

$$\beta_{11}' - \frac{\lambda(f' \cos\tau - f\sin\tau)\beta_{11}}{4F} = \frac{\alpha_{10}''}{2\,F^{\frac{1}{2}}} \tag{32a}$$

and

$$\alpha_{11}' - \frac{\lambda(f' \cos\tau - f\sin\tau)\alpha_{11}}{4F} = 0 \tag{32b}$$

Solving (32a, b) subject to (30a) gives

$$\beta_{11}(\tau) = (1 - \lambda f\cos\tau)^{-\frac{1}{4}}\left[\beta_{11}(0) - \frac{1}{2}\int_0^{\tau}\frac{\alpha_{11}''(s)ds}{F^{\frac{1}{4}}(s)}\right] \tag{33a}$$

$$\alpha_{11}(\tau) = 0 \tag{33b}$$

The remaining equation in (31) is now solved to get

$$\zeta^{12}(t,\tau) = \alpha_{12}(\tau)\cos t + \beta_{12}(\tau)\sin t - \frac{B''}{F}$$

$$\alpha_{12}(0) = \frac{B''(0)}{F}\,, \beta_{12}(0) = 0 \tag{34}$$

Meanwhile, we conclude from (33a, b) that

$$\zeta^{11}(t,\tau) = \beta_{11}(\tau)\sin t \tag{35}$$

and note that

$$B''(0) = \frac{\left[(1-\lambda)(1+\lambda^3) + 2\lambda^2 {f'}^2(0) + \lambda(1-\lambda)f''(0)\right]}{(1-\lambda)^3} \tag{36}$$

Substituting into (17) and simplifying gives

$$\zeta_{tt}^{30} + \zeta^{30} = \frac{2\mu_2'\alpha_{10}\cos t}{F^{\frac{1}{2}}} + \frac{b}{F}\left[\left(\frac{3B\alpha_{10}^2}{2} + B^3\right) + 3\left(\frac{\alpha_{10}^3}{4} + \alpha_{10}B^2\right)\cos t + \frac{3B\alpha_{10}^2\cos 2t}{2} + \frac{\alpha_{10}^3\cos 3t}{4}\right] \tag{37}$$

To ensure a uniformly valid solution in t, we equate to zero in (37) the coefficients of $\cos t$ and $\sin t$ and obtain

$$\mu_2'(\tau) = \frac{-3b\left(\frac{\alpha_{10}^2}{4} + B^2\right)}{2\,F^{\frac{1}{2}}} \tag{38a}$$

where

$$\mu_2'(0) = \frac{-15bB^2(0)}{8\,(1-\lambda)^{\frac{1}{2}}} \tag{38b}$$

The remaining equation in (37) is now solved to get

$$\zeta^{30}(t,\tau) = \alpha_{30}(\tau)\cos t + \beta_{30}(\tau)\sin t + \frac{b}{F}\left[\left(\frac{3B\alpha_{10}^2}{2} + B^3\right) - \frac{B\alpha_{10}^2\,\cos 2t}{2} - \frac{\alpha_{10}^3\cos 3t}{32}\right] \tag{39}$$

$$\alpha_{30}(0) = \frac{-65\,B^3(0)}{32\,(1-\lambda)}\,, \beta_{30}(0) = 0 \tag{40}$$

The following simplifications are necessary in the substitution into (18) which soon follows:

$$(\zeta^{10})^2 = \left(\frac{\alpha_{10}^2}{2} + B^2\right) + 2\alpha_{10}B\cos t + \frac{\alpha_{10}^2\,\cos 2t}{2}$$

$$(\zeta^{10})^2\zeta^{11} = \beta_{11}\left[\left(\frac{\alpha_{10}^2}{4} + B^2\right)\sin t + \alpha_{10}B\sin 2t + \frac{\alpha_{10}^2\,\cos 2t}{2} + \frac{\alpha_{10}^2\sin 3t}{4}\right] \tag{41}$$

Substituting into (18) and simplifying, we get

$$\zeta_{tt}^{31}+\zeta^{31}=\frac{2\mu_2'\beta_{11}\sin t}{F^{\frac{1}{2}}}+\frac{3b\beta_{11}}{F}\left[\left(\frac{\alpha_{10}^2}{4}+B^2\right)\sin t+\alpha_{10}B\sin 2t+\frac{\alpha_{10}^2\sin 3t}{4}\right]+\frac{2\mu_2'\alpha_{10}'\sin t}{F}$$
$$-\frac{2}{F^{\frac{1}{2}}}\left[-\alpha_{30}'\sin t+\beta_{30}'\cos t+\left(\frac{B\alpha_{10}^2}{F}\right)'\sin 2t+\frac{3}{32}\left(\frac{\alpha_{10}^3}{F}\right)'\sin 3t\right]$$
$$+\frac{\lambda(f'\cos\tau-f\sin\tau)}{2\,F^{\frac{1}{2}}}\left[-\alpha_{30}\sin t+\beta_{30}\cos t+\frac{b}{F}\left(B\alpha_{10}^2\sin 2t+\frac{3\alpha_{10}^3\sin 3t}{32}\right)\right]+\frac{\mu_2''\alpha_{10}\sin t}{F} \quad (42)$$

To ensure a uniformly valid solution in t, we equate to zero, in (42), the coefficients of $\cos t$ and $\sin t$ and respectively obtain:

$$\cos t:\ \frac{-2\beta_{30}'}{F^{\frac{1}{2}}}+\frac{\lambda(f'\cos\tau-f\sin\tau)\beta_{30}}{2\,F^{\frac{1}{2}}}=0 \quad (43)$$

$$\sin t:\ \frac{2\mu_2'\beta_{11}}{F^{\frac{1}{2}}}+\frac{3b\beta_{11}\left(\frac{\alpha_{10}^2}{4}+B^2\right)}{F}+\frac{2\alpha_{30}'}{F^{\frac{1}{2}}}+\frac{2\mu_2'\alpha_{10}'}{F}-\frac{\lambda(f'\cos\tau-f\sin\tau)\alpha_{30}}{2\,F^{\frac{3}{2}}}+\frac{\mu_2''\alpha_{10}}{F}=0 \quad (44)$$

Solving (43) and (44) easily yields

$$\beta_{30}(\tau)\equiv 0,\ \alpha_{30}(\tau)=(1-\lambda f\cos\tau)^{-\frac{1}{4}}\left[\int_0^{\tau}(1-\lambda f\cos\tau)^{\frac{1}{4}}\,H_1(s)ds+\alpha_{30}(0)\right] \quad (45)$$

where

$$H_1(\tau)=-\frac{1}{2}\left[2\mu_2'\beta_{11}+\frac{3b\beta_{11}\left(\frac{\alpha_{10}^2}{4}+B^2\right)}{F^{\frac{1}{2}}}+\frac{\mu_2''\alpha_{10}}{F^{\frac{1}{2}}}+\frac{2\mu_2'\alpha_{10}'}{F}\right] \quad (46)$$

The remaining equation in (42) is re-arranged as

$$\zeta_{tt}^{31}+\zeta^{31}=R_1(\tau)\sin 2t+R_2(\tau)\sin 3t \quad (47)$$

$$\zeta^{31}(0,0)=0,\ \zeta_t^{31}(0,0)+\frac{\mu_2'(0)\zeta_t^{11}(0,0)+\zeta_\tau^{30}(0,0)}{(1-\lambda)^{\frac{1}{2}}} \quad (48)$$

where

$$R_1=\frac{3b\beta_{11}B\alpha_{10}}{F}-\frac{2b\left(\frac{B\alpha_{10}^2}{F}\right)'}{F^{\frac{1}{2}}}+\frac{b\lambda(f'\cos\tau-f\sin\tau)B\alpha_{10}^2}{2F} \quad (49a)$$

$$R_2=\frac{3}{4}\left(\frac{b\alpha_{10}^2\beta_{11}}{F}\right)-\frac{3b}{16F}\left(\frac{\alpha_{10}^3}{F}\right)'+\frac{3b\lambda(f'\cos\tau-f\sin\tau)\alpha_{10}^3}{64F^{\frac{3}{2}}} \quad (49b)$$

$$R_1(0)=\frac{bB^3(0)f'(0)}{2(1-\lambda)}\left[\frac{3(4-\lambda)}{2(1-\lambda)^{\frac{3}{2}}}+\left(\lambda-\frac{2(2+3\lambda)}{(1-\lambda)^{\frac{3}{2}}}\right)\right] \quad (50a)$$

$$R_2(0)=\frac{3bB^3(0)f'(0)}{16(1-\lambda)^{\frac{5}{2}}}\left[\frac{7-\lambda\,(1-\lambda)^{\frac{1}{2}}}{4(1-\lambda)^{\frac{1}{2}}}-(4-\lambda)\right] \quad (50b)$$

Solving (47) – (48) gives

$$\zeta^{31}(t,\tau)=\alpha_{31}(\tau)\cos t+\beta_{31}(\tau)\sin t-\frac{R_1(\tau)\sin 2t}{3}-\frac{R_2(\tau)\sin 3t}{8} \quad (51)$$

where

$$\alpha_{31}(0)=0,\ \beta_{31}(0)+\frac{b}{(1-\lambda)^{\frac{1}{2}}}\left[\frac{1}{F}\left\{\alpha_{30}+\left(\frac{3B\alpha_{10}^2}{2}+B^3\right)-\frac{B\alpha_{10}^2}{2}-\frac{\alpha_{10}^3}{32}\right\}\right]'\Bigg|_{\tau=0}=0 \quad (52)$$

It is however worthy of note that from (38a)

$$\mu_2''(0)=\frac{3bB^2(0)f'(0)(8+3\lambda)}{8(1-\lambda)^{\frac{3}{2}}}. \quad (53)$$

The following simplification will be necessary in the substitution into (19) which follows shortly:

$$(\zeta^{10})^2\zeta^{12} = \left\{B\alpha_{10}\alpha_{12} - \frac{B''}{F}\left(\frac{\alpha_{10}^2}{4} + B^2\right)\right\} + \left\{\left(\frac{\alpha_{10}^2}{4} + B^2\right)\alpha_{12} + \frac{\alpha_{10}^2\,\alpha_{12}}{4} - \frac{2B''B\alpha_{10}}{F}\right\}\cos t + \left\{\beta_{12}\left(\frac{\alpha_{10}^2}{2} + B^2\right) - \frac{\beta_{12}\alpha_{10}^2}{4}\right\}\sin t + \left\{B\alpha_{10}\alpha_{12} - \frac{B''\alpha_{10}^2}{2F}\right\}\cos 2t + \beta_{12}B\alpha_{10}\sin 2t + \frac{\alpha_{10}^2\,\alpha_{12}}{4}\cos 3t + \frac{\alpha_{10}^2\,\beta_{12}}{4}\sin 3t \tag{54}$$

$$(\zeta^{11})^2\zeta^{10} = \frac{B\beta_{11}^2}{2} + \frac{\beta_{11}^2\alpha_{10}}{4}\cos t - \frac{B\beta_{11}^2}{2}\cos 2t - \frac{\beta_{11}^2\alpha_{10}}{4}\cos 3t \tag{55}$$

Substituting into (19) yields

$$\zeta_{tt}^{32} + \zeta^{32} = \frac{1}{F}\left[\alpha_{30}''\cos t + b\left\{\left(\frac{\frac{3}{2}B\alpha_{10}^2 + B^3}{F}\right)'' - \left(\frac{B\alpha_{10}^2}{2F}\right)''\cos 2t - \left(\frac{\alpha_{10}^3}{F}\right)''\cos 3t - \frac{2\mu_2'\beta_{11}'}{F}\cos t + \frac{2\mu_2'}{F^{\frac{1}{2}}}(\alpha_{12}\cos t + \beta_{12}\sin t)\right\}\right] - \frac{2}{F^{\frac{1}{2}}}\left[-\alpha_{31}'\sin t + \beta_{31}'\cos t - \frac{2R_1'\cos 2t}{3} - \frac{3R_2'\cos 3t}{8}\right] + \frac{\lambda(f'\cos\tau - f\sin\tau)}{2F^{\frac{3}{2}}}\left[-\alpha_{31}\sin t + \beta_{31}\cos t - \frac{2R_1\cos 2t}{3} - \frac{3R_2\cos 3t}{8}\right] - \frac{2\mu_2'\beta_{11}'\cos t}{F} - \frac{2\mu_2'\beta_{11}\cos t}{F} + \frac{3b}{F}\left\{B\alpha_{10}\alpha_{12} - \frac{B''}{F}\left(\frac{\alpha_{10}^2}{4} + B^2\right)\right\} + \left\{\left(\frac{\alpha_{10}^2}{4} + B^2\right)\alpha_{12} + \frac{\alpha_{10}^2\,\alpha_{12}}{4} - \frac{2B''B\alpha_{10}}{F}\right\}\cos t + \left\{\beta_{12}\left(\frac{\alpha_{10}^2}{2} + B^2\right) - \frac{\beta_{12}\alpha_{10}^2}{4}\right\}\sin t + \left\{B\alpha_{10}\alpha_{12} - \frac{B''\alpha_{10}^2}{2F}\right\}\cos 2t + \beta_{12}B\alpha_{10}\sin 2t + \frac{\alpha_{10}^2\,\alpha_{12}}{4}\cos 3t + \frac{\alpha_{10}^2\,\beta_{12}}{4}\sin 3t + \frac{B\beta_{11}^2}{2} + \frac{\beta_{11}^2\alpha_{10}}{4}\cos t - \frac{B\beta_{11}^2}{2}\cos 2t - \frac{\beta_{11}^2\alpha_{10}}{4}\cos 3t \tag{56}$$

$$\zeta^{32}(0,0) = 0,\ \zeta_t^{32}(0,0) + \frac{\mu_2'(0)\zeta_t^{12}(0,0) + \zeta_\tau^{31}(0,0)}{(1-\lambda)^{\frac{1}{2}}} = 0 \tag{57}$$

To ensure a uniformly valid solution in t, we equate to zero, in (56), the coefficients of $\cos t$ and $\sin t$ and respectively obtain:

$$\cos t: \frac{\alpha_{30}''}{F} + \frac{2\mu_2'\beta_{11}'b}{F} - \frac{2\mu_2'\alpha_{12}b}{F^{\frac{3}{2}}} - \frac{2\beta_{31}'}{F^{\frac{1}{2}}} + \frac{\lambda(f'\cos\tau - f\sin\tau)\beta_{31}}{2F^{\frac{3}{2}}} - \frac{\mu_2'\beta_{11}'}{F} + \frac{3b}{F}\left\{\left(\frac{\alpha_{10}^2}{2} + B^2\right)\alpha_{12} + \frac{\alpha_{10}^2\,\alpha_{12}}{4} - \frac{2B''B\alpha_{10}}{F}\right\} + \frac{\beta_{11}^2\alpha_{10}}{4} = 0 \tag{58}$$

$$\sin t: \frac{2\alpha_{31}}{F^{\frac{1}{2}}} - \frac{\lambda(f'\cos\tau - f\sin\tau)\alpha_{31}}{2F^{\frac{3}{2}}} + \frac{3b\beta_{12}}{F}\left(\frac{\alpha_{10}^2}{4} + B^2\right) = 0 \tag{59}$$

Solving (58), we get

$$\beta_{31}(\tau) = (1-\lambda f\cos\tau)^{-\frac{1}{4}}\left[\int_0^\tau (1-\lambda f\cos\tau)^{\frac{1}{4}}H_2(s)ds + \beta_{31}(0)\right] \tag{60}$$

where

$$H_2(\tau) = \frac{F^{\frac{1}{2}}}{2}\left[\frac{3b}{F}\left\{\left(\frac{\alpha_{10}^2}{2} + B^2\right)\alpha_{12} + \frac{\alpha_{10}^2\,\alpha_{12}}{4} - \frac{2B''B\alpha_{10}}{F}\right\} + \frac{\beta_{11}^2\alpha_{10}}{4} + \frac{\mu_2'\beta_{11}'b}{F} - \frac{2\mu_2'\alpha_{12}b}{F^{\frac{3}{2}}} - \frac{\mu_2''B}{F}\right] \tag{61}$$

Similarly, the solution to (59) is

$$\alpha_{31}(\tau) = (1-\lambda f\cos\tau)^{-\frac{1}{4}}\left[\int_0^\tau (1-\lambda f\cos\tau)^{\frac{1}{4}}H_3(s)ds + \beta_{31}(0)\right] \tag{62}$$

$$H_3(\tau) = \frac{3b\beta_{12}}{2F^{\frac{1}{2}}}\left(\frac{\alpha_{10}^2}{2} + B^2\right) \tag{63}$$

The remaining equation in (56) is now arranged as

$$\zeta_{tt}^{32} + \zeta^{32} = R_3(\tau) + R_4\cos 2t + R_5\sin 2t + R_6\cos 3t + R_7\sin 3t \tag{64}$$

where

$$R_3 = -\frac{b}{F}\left(\frac{\frac{3B\alpha_{10}^2}{2} + B^3}{F}\right)'' + \frac{3b}{F}\left\{B\alpha_{10}\alpha_{12} - \frac{B''}{F}\left(\frac{\alpha_{10}^2}{4} + B^2\right)\right\} \tag{65a}$$

$$R_4 = \frac{b}{2F}\left(\frac{B\alpha_{10}^2}{F}\right)'' + \frac{R_1'}{3F^{\frac{1}{2}}} - \frac{2\lambda(f'\cos\tau - f\sin\tau)R_1}{3F^{\frac{3}{2}}} + \frac{3b}{F}\left(B\alpha_{10}\alpha_{12} - \frac{B''\alpha_{10}^2}{2F}\right) - \frac{3bB\beta_{11}^2}{2F} \quad (65b)$$

$$R_5 = \frac{3bB\alpha_{10}\beta_{12}}{F} \quad (65c)$$

$$R_6 = \frac{b}{F}\left(\frac{\alpha_{10}^2}{F}\right)'' + \frac{3R_2'}{4F^{\frac{1}{2}}} - \frac{3\lambda(f'\cos\tau - f\sin\tau)R_2}{8F^{\frac{3}{2}}} + \frac{3b\alpha_{10}^2\alpha_{12}}{F} - \frac{3b\alpha_{10}\beta_{11}^2}{4F} \quad (65d)$$

$$R_7 = \frac{3b\beta_{12}\alpha_{10}^2}{4F} \quad (65e)$$

On solving (64) with (65a – e), we have

$$\zeta^{32}(t,\tau) = \alpha_{32}(\tau)\cos t + \beta_{32}(\tau)\sin t + R_3 - \frac{R_4\cos 2t + R_5\sin 2t}{3} - \frac{R_6\cos 3t + R_7\sin 3t}{8} \quad (66a)$$

$$\alpha_{32}(0) = -R_3(0) - \frac{R_4(0)}{3} - \frac{R_6(0)}{8} \quad (66b)$$

$$\beta_{32}(0) = -\frac{2R_5(0)}{3} - \frac{3R_7(0)}{8} + \frac{\mu_2'(0)\zeta_t^{12}(0,0) + \zeta_\tau^{31}(0,0)}{(1-\lambda)^{\frac{3}{2}}} = 0 \quad (66c)$$

We will also make use of the following terms. From (38a),

$$\mu_2''(0) = \frac{-3bB^2(0)f'(0)}{2(1-\lambda)}R_8(0) \quad (67a)$$

$$R_8(0) = (1-\lambda)^{\frac{1}{2}}\left(\frac{B(0)}{8} + \frac{2}{(1-\lambda)}\right) + \frac{5\lambda}{(1-\lambda)^{\frac{1}{2}}} \quad (67b)$$

By substituting into (46), we get

$$H_1(0) = -\frac{1}{2}\left[2\mu_2'\beta_{11}(0) + \frac{15b\beta_{11}(0)B^3(0)}{4(1-\lambda)^{\frac{1}{2}}} - \frac{B(0)\mu_2''}{(1-\lambda)^{\frac{1}{2}}} + \frac{2\mu_2'\alpha_{10}'}{(1-\lambda)^{\frac{1}{2}}}\right] \quad (68)$$

Evaluating (44) at zero gives

$$\alpha_{30}'(0) = -\frac{(1-\lambda)^{\frac{1}{2}}}{2}\left[\frac{2\mu_2'(0)\beta_{11}(0)}{(1-\lambda)^{\frac{1}{2}}} + \frac{15b\beta_{11}(0)B^3(0)}{4(1-\lambda)} + \frac{2\mu_2'(0)\alpha_{10}'(0)}{(1-\lambda)} - \frac{\mu_2''(0)B(0)}{(1-\lambda)}\right] + \frac{\lambda f'(0)\alpha_{30}(0)}{4(1-\lambda)} \quad (69)$$

Following (13), we can summarize the analysis so far as

$$\eta(t,\tau) = \bar{\xi}(\zeta^{10} + \delta\zeta^{11} + \delta^2\zeta^{12} + \cdots) + \bar{\xi}^3(\zeta^{30} + \delta\zeta^{31} + \delta^2\zeta^{32} + \cdots) + \cdots \quad (70)$$

Values of Variables at maximum Displacement

At maximum displacement η_a, we let the values of $t, \hat{t}$ and τ to be $t_a, \hat{t}_a$ and τ_a respectively and let them be expanded asymptotically as

$$t_a = t_0 + t_{01}\delta + t_{02}\delta^2 + \cdots \bar{\xi}^2(t_{20} + t_{21}\delta + t_{22}\delta^2 + \cdots) + \cdots \quad (71a)$$

$$\hat{t}_a = \hat{t}_0 + \hat{t}_{01}\delta + \hat{t}_{02}\delta^2 + \cdots \bar{\xi}^2(\hat{t}_{20} + \hat{t}_{21}\delta + \hat{t}_{22}\delta^2 + \cdots) + \cdots \quad (71b)$$

$$\tau_a = \delta\hat{t}_a = \delta\left[\hat{t}_0 + \hat{t}_{01}\delta + \hat{t}_{02}\delta^2 + \cdots \bar{\xi}^2(\hat{t}_{20} + \hat{t}_{21}\delta + \hat{t}_{22}\delta^2 + \cdots)\right] + \cdots \quad (71c)$$

The condition for maximum displacement follows from (9) as

$$\eta_t + (1-\lambda f\cos\tau)^{-\frac{1}{2}}\left[(\mu_2'\bar{\xi}^2 + \mu_2'\bar{\xi}^2 + \cdots)\eta_t + \delta\eta_\tau\right] = 0 \quad (72)$$

Now, the expansion of each of the terms in (72), using (70) and (71a, b, c) follows as:

$$\bar{\xi}\zeta_t^{10} = \bar{\xi}\,[\zeta_t^{10} + \{t_{01}\delta + t_{02}\delta^2 + \cdots + \bar{\xi}^2(t_{20} + t_{21}\delta + \cdots)\}\zeta_{tt}^{10} + \delta\{\hat{t}_0 + \hat{t}_{01}\delta + \cdots$$
$$+ \bar{\xi}^2(\hat{t}_{20} + \hat{t}_{21}\delta + \cdots) + \cdots\}\zeta_{t\tau}^{10} + \frac{1}{2}\{t_{01}\delta + t_{02}\delta^2 + \cdots + \bar{\xi}^2(t_{20} + t_{21}\delta + \cdots)\}^2\zeta_{ttt}^{10}$$

$$+\delta^2\{\hat{t}_{01}\delta+\hat{t}_{02}\delta^2+\cdots+\bar{\xi}^2(\hat{t}_{20}+\hat{t}_{21}\delta+\cdots)\}\zeta_{t\tau}^{10}+2\delta\{t_{01}\delta+t_{02}\delta^2+\cdots$$
$$+\bar{\xi}^2(t_{20}+t_{21}\delta+\cdots)\}\{\hat{t}_0+\hat{t}_{01}\delta+\cdots+\bar{\xi}^2(\hat{t}_{20}+\hat{t}_{21}\delta+\cdots)\}\zeta_{tt\tau}^{10}]+\cdots \quad (73)$$

$$\bar{\xi}\delta\zeta_t^{11}=\bar{\xi}\delta[\zeta_t^{11}+\{t_{01}\delta+t_{02}\delta^2+\cdots+\bar{\xi}^2(t_{20}+t_{21}\delta+\cdots)\}\zeta_{tt}^{11}+\delta\{\hat{t}_0+\hat{t}_{01}\delta+\cdots$$
$$+\bar{\xi}^2(\hat{t}_{20}+\hat{t}_{21}\delta+\cdots)+\cdots\}\zeta_{t\tau}^{11}+\cdots] \quad (74)$$

$$\bar{\xi}^3\zeta_t^{30}=\bar{\xi}^3[\zeta_t^{30}+\{t_{01}\delta+t_{02}\delta^2+\cdots+\bar{\xi}^2(t_{20}+t_{21}\delta+\cdots)\}\zeta_{tt}^{30}+\delta\{\hat{t}_0+\hat{t}_{01}\delta+\cdots$$
$$+\bar{\xi}^2(\hat{t}_{20}+\hat{t}_{21}\delta+\cdots)+\cdots\}\zeta_{t\tau}^{30}+\cdots] \quad (75)$$

$$\bar{\xi}^3\delta\zeta_t^{31}=\bar{\xi}^3\delta[\zeta_t^{31}+\{t_{01}\delta+t_{02}\delta^2+\cdots+\bar{\xi}^2(t_{20}+t_{21}\delta+\cdots)\}\zeta_{tt}^{31}+\delta\{\hat{t}_0+\hat{t}_{01}\delta+\cdots$$
$$+\bar{\xi}^2(\hat{t}_{20}+\hat{t}_{21}\delta+\cdots)+\cdots\}\zeta_{t\tau}^{31}+\cdots] \quad (76)$$

$$(1-\lambda f\cos\tau)^{-\frac{1}{2}}\bar{\xi}^3\mu_2'\zeta_t^{10}=\bar{\xi}^3\Big[(1-\lambda)^{-\frac{1}{2}}\mu_2'\zeta_t^{10}+(1-\lambda)^{-\frac{1}{2}}\mu_2'(0)\{t_{01}\delta+\cdots\}\zeta_{tt}^{10}+\cdots$$
$$+\delta\{\hat{t}_0+\hat{t}_{01}\delta+\cdots\}\Big((1-\lambda f\cos\tau)^{-\frac{1}{2}}\mu_2'\zeta_t^{10}\Big)_\tau+\cdots\Big] \quad (77)$$

$$(1-\lambda f\cos\tau)^{-\frac{1}{2}}\bar{\xi}^3\mu_2'\zeta_t^{11}=\bar{\xi}^3\delta\Big[(1-\lambda)^{-\frac{1}{2}}\mu_2'\zeta_t^{11}+(1-\lambda)^{-\frac{1}{2}}\mu_2'(0)\{t_{01}\delta+\cdots\}\zeta_{tt}^{11}+\cdots$$
$$+\delta\{\hat{t}_0+\hat{t}_{01}\delta+\cdots\}\Big((1-\lambda f\cos\tau)^{-\frac{1}{2}}\mu_2'\zeta_t^{11}\Big)_\tau+\cdots\Big] \quad (78)$$

$$(1-\lambda f\cos\tau)^{-\frac{1}{2}}\bar{\xi}\delta\zeta_\tau^{10}=\bar{\xi}\delta\Big[(1-\lambda)^{-\frac{1}{2}}\zeta_\tau^{10}+(1-\lambda)^{-\frac{1}{2}}\{t_{01}\delta+\cdots\}\zeta_{t\tau}^{10}+\cdots$$
$$+\delta\{\hat{t}_{01}\delta+\cdots+\bar{\xi}^2(\hat{t}_{20}+\cdots)\}\left(\frac{\zeta_\tau^{10}}{(1-\lambda f\cos\tau)^{\frac{1}{2}}}\right)_\tau+\frac{1}{2}\left\{\{t_{01}\delta+\cdots+\bar{\xi}^2(t_{20}+\cdots)\}^2\frac{\zeta_{\tau tt}^{10}}{(1-\lambda)^{\frac{1}{2}}}+\cdots\right\}\Big] \quad (79)$$

$$(1-\lambda f\cos\tau)^{-\frac{1}{2}}\bar{\xi}\delta^2\zeta_\tau^{11}=\bar{\xi}\delta^2\Big[(1-\lambda)^{-\frac{1}{2}}\zeta_\tau^{11}+(1-\lambda)^{-\frac{1}{2}}\{t_{01}\delta+\cdots\}\zeta_{t\tau}^{11}+\cdots\Big] \quad (80)$$

$$(1-\lambda f\cos\tau)^{-\frac{1}{2}}\bar{\xi}^3\delta\zeta_\tau^{30}=\bar{\xi}\delta^2\Big[(1-\lambda)^{-\frac{1}{2}}\zeta_\tau^{30}+(1-\lambda)^{-\frac{1}{2}}\{t_{01}\delta+\cdots\}\zeta_{t\tau}^{30}+\cdots\Big] \quad (81)$$

$$(1-\lambda f\cos\tau)^{-\frac{1}{2}}\bar{\xi}^3\delta^2\zeta_\tau^{31}=\bar{\xi}\delta^2\Big[(1-\lambda)^{-\frac{1}{2}}\zeta_\tau^{31}+(1-\lambda)^{-\frac{1}{2}}\{t_{01}\delta+\cdots\}\zeta_{t\tau}^{31}+\cdots\Big] \quad (82)$$

where (73) – (82) are evaluated at $(t_0,0)$.

Now, substituting from (73) – (82) into (72) and equating the coefficients of $\bar{\xi}^i\delta^j$, we have:

$$O(\bar{\xi}):\ \zeta_t^{10}=0 \quad (83)$$

$$O(\bar{\xi}\delta):\ t_{01}\zeta_{tt}^{10}+\hat{t}_0\zeta_{t\tau}^{10}+\zeta_t^{11}+\frac{\zeta_\tau^{10}}{(1-\lambda)^{\frac{1}{2}}}=0, \quad (84)$$

$$O(\bar{\xi}^3):\ t_{20}\zeta_{tt}^{10}+\zeta_t^{30}+\frac{\mu_2'(0)\zeta_t^{10}}{(1-\lambda)^{\frac{1}{2}}}=0, \quad (85)$$

$$O(\bar{\xi}^3\delta):\ t_{21}\zeta_{tt}^{10}+\hat{t}_{20}\zeta_{t\tau}^{10}+t_{20}\zeta_{tt}^{11}+t_{01}\zeta_{tt}^{30}+\hat{t}_0\zeta_{t\tau}^{30}+\zeta_t^{31}+t_{01}t_{20}\zeta_{ttt}^{10}+\frac{\mu_2'(0)t_{01}\zeta_{tt}^{10}}{(1-\lambda)^{\frac{1}{2}}}$$
$$+\hat{t}_0\left[\frac{\mu_2'\zeta_t^{10}}{(1-\lambda f\cos\tau)^{\frac{1}{2}}}\right]_\tau+\frac{\mu_2'(0)\zeta_t^{11}+t_{20}\zeta_{t\tau}^{10}+\zeta_\tau^{30}}{(1-\lambda)^{\frac{1}{2}}}=0 \quad (86)$$

From (83), using (29), we get

$$\sin t_0=0\ \Rightarrow\ t_0=n\pi, n=1,2,3,\ldots$$

We take the least nontrivial value of t_0, for the case $n=1$, and get

$$t_0=\pi \quad (87)$$

Substituting into (84), noting that $\zeta_{tt}^{10}(t,0)=0$ we get

$$t_{01}=-\frac{\zeta_t^{11}+(1-\lambda)^{-\frac{1}{2}}\zeta_\tau^{10}}{\zeta_{tt}^{10}}=\frac{f'(0)}{4(1-\lambda)}\left[\frac{4-\lambda}{(1-\lambda)^{\frac{1}{2}}}+(4+\lambda)\right] \quad (88)$$

Substituting into (85) easily yields

$$t_{20} = 0 \tag{89}$$

On substitution, most terms in (86) vanish, leaving

$$t_{21} = -\frac{1}{\zeta_{tt}^{10}}\left[t_{01}\zeta_{tt}^{30} + \zeta_{t}^{31} + \frac{\mu_2'(0)\, t_{01}\zeta_{tt}^{10} + \mu_2'(0)\zeta_t^{11} + \zeta_\tau^{30}}{(1-\lambda)^{\frac{1}{2}}}\right]\Bigg|_{(t_0,0)} \tag{90}$$

etc.
From (70), the non-vanishing terms, in the maximum displacement $\eta(t_a, \tau_a)$ can be expanded to get

$$\eta_a = \eta(t_a, \tau_a) = \left[\bar{\xi}\,(\zeta^{10} + \delta\hat{t}_0\zeta_\tau^{10} + \cdots) + \bar{\xi}^3\,\{\zeta^{30} + \delta(\hat{t}_{20}\zeta_\tau^{10} + \hat{t}_0\zeta_\tau^{30})\} + \cdots\right]\Big|_{(t_0,0)} \tag{91}$$

We note that

$$\zeta_\tau^{30}(t_0, 0) = \frac{bB^3(0)f'(0)}{128(1-\lambda)^2}(496 - 179\lambda) \tag{92a}$$

$$\zeta_\tau^{10}(t_0, 0) = \frac{B(0)f'(0)(4+\lambda)}{4(1-\lambda)} \tag{92b}$$

$$\zeta^{30}(t_0, 0) = \frac{4bB^3(0)}{(1-\lambda)} \tag{92c}$$

Substituting (92a, b, c) into (91) and simplifying yields

$$\eta_a = 2B(0)\bar{\xi}\left[1 + \frac{\delta f'(0)(4+\lambda)}{8(1-\lambda)} + \cdots\right] + \frac{4bB^3(0)\bar{\xi}^3}{(1-\lambda)}\left[1 + \frac{\delta f'(0)}{16}\left\{\frac{\hat{t}_{20}(4+\lambda)}{B^2(0)} + \frac{\hat{t}_0(496-179\lambda)}{32}\right\} + \cdots\right] \tag{93}$$

which can further be written as

$$\eta_a = \bar{\xi} f_1 + \bar{\xi}^3 f_3 + \cdots \tag{94a}$$

$$f_1 = 2B(0)(1 + A_1 f'(0)\delta + \cdots) \tag{94b}$$

$$A_1 = \frac{(4+\lambda)}{8(1-\lambda)} \tag{94c}$$

$$f_3 = \frac{4bB(0)}{(1-\lambda)}(1 + A_3 f'(0)\delta + \cdots) \tag{94d}$$

$$A_3 = \frac{1}{16}\left[\frac{\hat{t}_{20}(4+\lambda)}{B^2(0)} + \frac{\hat{t}_0(496-179\lambda)}{32}\right] + \cdots \tag{94e}$$

So far, the terms $\hat{t}_{20}$ and $\hat{t}_0$ appearing in (93) – (94e) are yet to be determined and are now determined by recourse to expansions of (5) and (6) as determined at their critical values. Thus, we have

$$\begin{aligned}\frac{d\bar{t}}{d\hat{t}} &= (1-\lambda f(\delta\hat{t})\cos\delta\hat{t})^{\frac{1}{2}} = (1-\lambda)^{\frac{1}{2}}\left[1 - \frac{\lambda}{2(1-\lambda)}\left\{f'(0)\delta\hat{t} + \frac{1}{2}(f''(0)-1)(\delta\hat{t})^2\right\}\right.\\ &\quad -\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\{f'(0)(\delta\hat{t})^2 + f'(0)(f''(0)-1)(\delta\hat{t})^3 + \cdots\} + \cdots]\\ &= (1-\lambda)^{\frac{1}{2}}\left[1 - \frac{\lambda f'(0)\delta\hat{t}}{2(1-\lambda)} - \left\{\frac{\lambda(f''(0)-1)}{4(1-\lambda)} + \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2 f'^2(0)\right\}(\delta\hat{t})^2\right.\\ &\quad \left. -\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2 f'(0)(f''(0)-1)(\delta\hat{t})^3 + \cdots\right]\end{aligned} \tag{95}$$

Thus, integrating (95) and evaluating it at the critical values of $\bar{t}$ and $\hat{t}$, we get

$$\begin{aligned}\bar{t}_a &= (1-\lambda)^{\frac{1}{2}}\left[\hat{t}_a - \frac{\lambda f'(0)\delta\hat{t}^2}{4(1-\lambda)} - \frac{1}{3}\left\{\frac{\lambda(f''(0)-1)}{4(1-\lambda)} + \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2 f'^2(0)\right\}\delta^2\hat{t}_a^3\right.\\ &\quad \left. -\frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2 f'(0)(f''(0)-1)\delta^3\hat{t}_a^4\right]\end{aligned} \tag{96}$$

Using (101a,b,c), we next expand both sides of (96) and thereafter equate the coefficients of $O(1)$ and $\bar{\xi}^2$ to get respectively

$$\bar{t}_0 = (1-\lambda)^{\frac{1}{2}}\hat{t}_0,\ \bar{t}_{20} = (1-\lambda)^{\frac{1}{2}}\hat{t}_{20} \tag{97}$$

Next, we determine (6) at the critical values followed by appropriate expansion using (71a)- (71c) and also use the fact that $\mu_i(0) = 0, i = 2, 3, ...$ after which we equate the coefficients of $O(1)$ and $\bar{\xi}^2$ in the expansion and get

$$t_0 = \bar{t}_0 = \pi, \text{ (using (87))} \tag{98}$$

$$t_{20} = 0 = \bar{t}_{20} + \mu_2'(0)\hat{t}_0 \text{ , (using (89))}$$

$$\therefore \quad \bar{t}_{20} = \mu_2'(0)\hat{t}_0 \tag{99}$$

From the first of (97) and (98), it follows that

$$\hat{t}_0 = (1-\lambda)^{-\frac{1}{2}}\pi \tag{100a}$$

While from the second of (97) and (99), we get

$$\hat{t}_{20} = -\mu_2'(0)(1-\lambda)^{-\frac{1}{2}}\hat{t}_0 \tag{100b}$$

for $\mu_2'(0)$ as in (38b). Hence, we have fully determined $\hat{t}_0$ and $\hat{t}_{20}$.

Dynamic Buckling Load

The dynamic buckling load, λ_D is defined as the highest value of the load parameter for the solution to be bounded and is determined by recourse to (3) which now takes the equivalent form

$$\frac{d\lambda}{d\eta_a} = 0$$

for η_a as in (92) or (93a).

As in Ozoigbo *et al* [7], the procedure is to first reverse the series (93a) in the form

$$\bar{\xi} = l_1\eta_a + l_3\eta_a^3 + \cdots \tag{101}$$

By substituting for η_a form (103a) in (101) and equating the coefficients of $\bar{\xi}$, we get

$$l_1 = \frac{1}{f_1},\ \ l_3 = -\frac{f_3}{f_1^4} \tag{102}$$

The maximization $\frac{d\lambda}{d\eta_a} = 0$ follows to give

$$\eta_{aD} = \sqrt{\frac{f_1^3}{3f_3}} \tag{103}$$

where, $\eta_{aD} = \eta_a(D)$ is the value of η_a at dynamic buckling. On substituting into (103) and simplifying, we get

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}}{2}(b)^{\frac{1}{2}}\lambda_D\bar{\xi}\ \left[\frac{1+A_3f'(0)\delta}{1+A_1f'(0)\delta}\right]^{\frac{1}{2}} \tag{104}$$

which determines the dynamic buckling load λ_D.

Analysis of Result

The result (104) is asymptotic in nature and so improves significantly according to the smallness of the magnitudes of $\bar{\xi}$ and δ. It is easily observed that the result also depends on the first derivative of load function evaluated at the initial time. This dependence on derivative of the load depends on the degree of accuracy retained in the determination of the maximum displacement η_a as in (91) because, if we had retained up to the terms in δ^2 in (91), the result would have depended on both $f'(0)$ and $f''(0)$ as well as on the squares of $f'(0)$. For a step load, $\cos\tau \equiv 1$ and $f(\tau) \equiv 1$, so that from (104),

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}}{2}(b)^{\frac{1}{2}}\lambda_D\bar{\xi} \tag{105}$$

Which was originally obtained by Budiansky [20] using Phase Plane analysis. It is to be stressed perhaps that the novelty in this analysis is that the load function depends explicitly on the time variable and that the method of approach is strictly analytical and not numerical.

The graph comparing Equation (105) and (104) is presented below as Graph 1 and Graph 2 respectively.

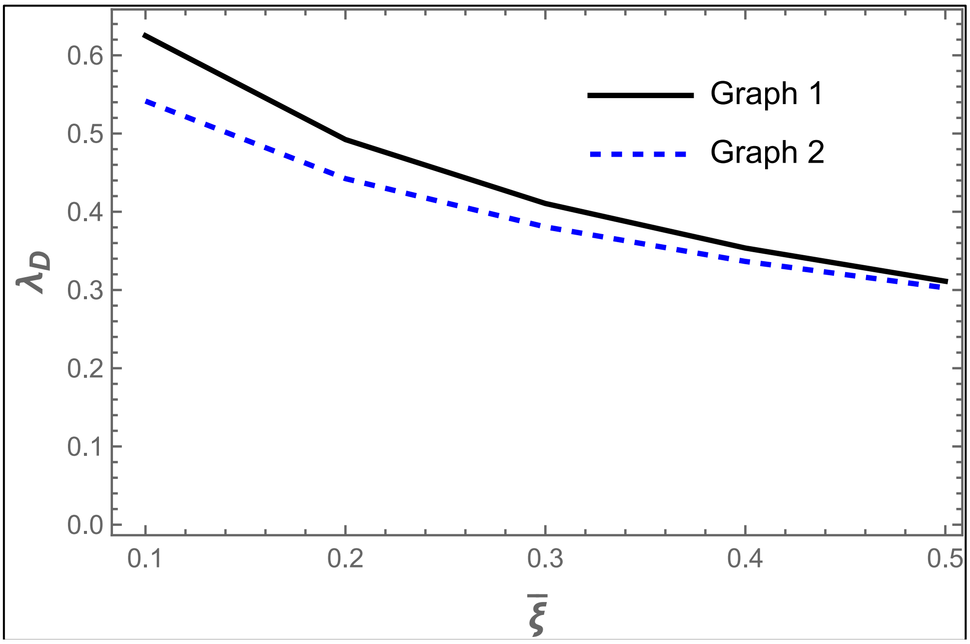

Fig.2: Dynamic buckling load λ_D versus imperfection parameter $\bar{\xi}$. Graph 1, is for step load using Eqn. (105) and Graph 2 is obtained using Eqn. (104)

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