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Characterization and estimation of length biased Quasi Shanker distribution with applications of real-life data

Dr. S Arunkumar and M Karthik

Abstract
In this paper, we have introduced a length biased version of quasi Shanker distribution called as the length biased quasi Shanker distribution. The length biased distribution is a special case of weighted distribution. The newly proposed distribution addresses with different statistical properties such as order statistics, Entropies, moment generating function and likelihood ratio test. The method of maximum likelihood estimation is also used for estimating the parameters of the proposed distribution. Finally, the newly proposed distribution is demonstrated with an application of two real life data sets to illustrate its usefulness and superiority.

Keywords: Weighted distribution, Quasi Shanker distribution, maximum likelihood estimation, order statistics, reliability analysis

1. Introduction

The weighted distributions are used when an investigator records an observation by nature according to certain stochastic model. The study of weighted distributions are useful in distribution theory because it provides a new understanding of the existing standard probability distributions and it provides methods for extending existing standard probability distributions for modelling lifetime data due to introduction of additional parameter in the model which creates flexibility in their nature. Weighted probability models plays an important role in some situations arising in various practical fields like medical sciences, engineering etc. These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non-experimental, non-replicated and non-random categories. Weighted distributions are required when the recorded observation from an event cannot randomly sample from actual distribution. This happens when the original observation damaged as well as an event occur in non-observability manner. Due to these inappropriate situations, resulting values are reduced, and units or events do not have same chances of occurrences as if they follow the exact distribution. The weighted distributions are applied in various research areas related to biomedicine, reliability, ecology and branching processes. In many applied sciences like engineering, medicine, behavioural science, finance, insurance and others, it is very crucial to modelling and analyzing lifetime data. For modelling this type of lifetime data, a number of continuous distributions are for modelling like weibull, Lindsey, exponential, lognormal and gamma. Let the original observation x has pdf $f(x)$ then in case of any biased in sampling appropriate weighted function, say $w(x)$ which is a function of random variable will be introduced to model the situation. This concept of weighted distributions was given by Fisher (1934) [4] to model the ascertainment bias. Later Rao (1965) [12] developed this concept in a unified manner while modelling the statistical data when the standard distributions were not appropriate to record these observations with equal probabilities. As a result, weighted models were formulated in such situations to record the observations according to some weighted function. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. The concept of length biased sampling was first introduced by Cox (1969) [3] and Zelen (1974) [21]. More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. There are various good sources which provide the detailed description of weighted distributions.

Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different fields. Afaq *et al* (2016) [2] have obtained the length biased weighted version of lomax distribution with properties and applications. Rather and Subramanian (2018) [15] discussed the characterization and estimation of length biased weighted generalized uniform distribution. Reyad *et al*. (2017) [13], obtained the length biased weighted frechet distribution with properties and estimation. Mudasir and Ahmad (2018) [10], discussed the characterization and estimation of length biased Nakagami distribution. Reyad *et al*. (2017) [13] have obtained the length biased weighted erlang distribution. Hassan *et al*. (2019, 2018a, 2018b) [9, 7, 8] introduced three weighted probability models with applications in handling data sets from engineering and medical sciences. Para and Jan (2018) [11] introduced three parameter weighted Pareto type II distribution with properties and applications in medical sciences. Rather and Ozel (2020) [18] introduced the weighted power lindley distribution with application of life time data. Rajagopalan, Ganaie and Rather (2019) [16] have discussed the length biased Aradhana distribution with applications. Recently, Ganaie and Rajagopalan (2021) [6] also discussed the length biased two parameter Pranav distribution with applications of real life time data, which shows more flexible and reliable than the classical distribution.
In this paper, we have considered the length biased version of quasi Shanker distribution known as length biased quasi shanker distribution. The quasi Shanker distribution is a newly proposed two parametric probability distribution formulated by shanker (2017) [19] and discussed its various structural properties including its moments and moment based measures, hazard rate function, stochastic ordering, stress-strength reliability and Bonferroni and Lorenz curves. The two parametric quasi Shanker distribution is a particular case of one parameter Shanker distribution. The proposed two parametric quasi Shanker distribution has better flexibility in data handling over one parameter exponential and lindley distribution.

2. Length Biased Quasi Shanker (LBQS) Distribution
The probability density function of quasi Shanker distribution is given by

$$f(x;\theta,\alpha)=\frac{\theta^3}{\theta^3+\theta+2\alpha}\left(\theta+x+\alpha x^2\right)e^{-\theta x};\ x>0,\theta>0,\theta^3+\theta+2\alpha>0$$

and the cumulative distribution function of the quasi Shanker distribution is given by

$$F(x;\theta,\alpha)=1-\left(1+\frac{\alpha\theta^2x^2+\theta x(\theta+2\alpha)}{\theta^3+\theta+2\alpha}\right)e^{-\theta x};x>0,\theta>0$$

Suppose X is a non-negative random variable with probability density function $f(x)$. aLet $w(x)$ be the non-negative weight function, then, the probability density function of the weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\ x>0.$$

Where $w(x)$ be the non - negative weight function and $E(w(x))=\int w(x)f(x)dx<\infty.$

In this Paper, we have to obtain the length biased version of quasi Shanker distribution called as length biased quasi Shanker distribution. We should note that we have different choices of weighted function *w(x)* gives different weighted models. Consequently, when *w(x)* = *x*, the resulting distribution is called length biased distribution and the probability density function of length biased quasi Shanker distribution is given by

$$f_l(x;\theta,\alpha)=\frac{xf(x;\theta,\alpha)}{E(x)},\quad x>0 \tag{3}$$

Where $E(x)=\int_0^\infty xf(x;\theta,\alpha)dx$

$$E(x)=\frac{\theta^3+2\theta+6\alpha}{\theta(\theta^3+\theta+2\alpha)} \tag{4}$$

Substitute equations (1) and (4) in equation (3), we will obtain the probability density function of length biased quasi Shanker distribution

$$f_l(x;\theta,\alpha)=\frac{x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}}{(\theta^3+2\theta+6\alpha)} \tag{5}$$

and the cumulative distribution function of length biased quasi Shanker distribution is obtained as

$$F_l(x;\theta,\alpha)=\int_0^x f_l(x;\theta,\alpha)dx$$

$$F_l(x;\theta,\alpha)=\int_0^x \frac{x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}}{(\theta^3+2\theta+6\alpha)}dx$$

$$F_l(x;\theta,\alpha)=\frac{1}{(\theta^3+2\theta+6\alpha)}\int_0^x x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}dx$$

Put $\theta x = t$, $\theta dx = dt$, As $x \to 0, t \to 0$ and $x \to x$, $t \to \theta x$
After simplification, we obtain the cumulative distribution function of length biased quasi Shanker distribution as

$$F_l(x;\theta,\alpha)=\frac{1}{(\theta^3+2\theta+6\alpha)}\left(\theta^3\gamma(2,\theta x)+\theta\gamma(3,\theta x)+\alpha\gamma(4,\theta x)\right)$$

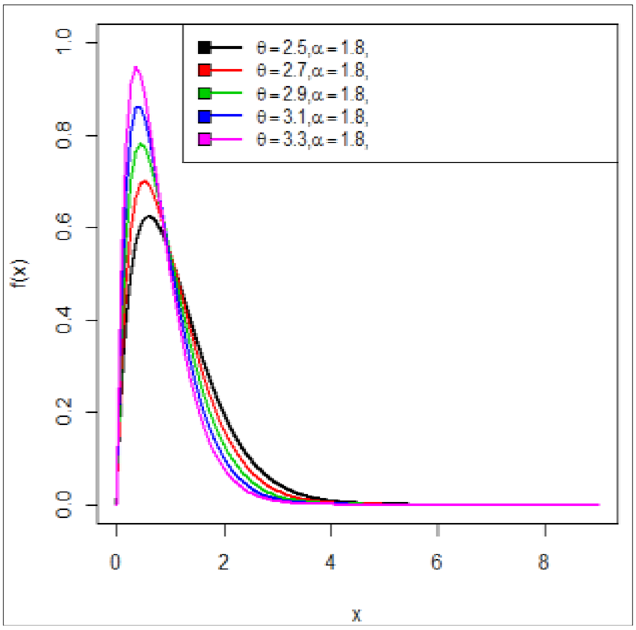

Fig 1: PDF of LBQS distribution

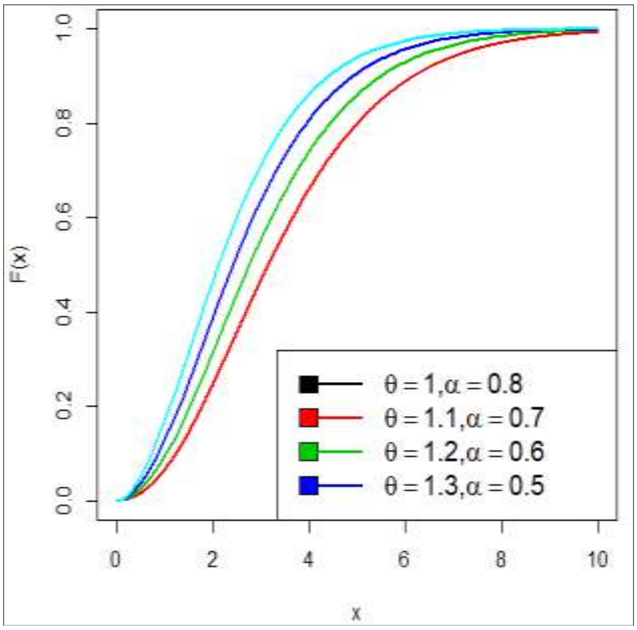

Fig 2: CDF plot of LBQS distribution

3. Reliability Analysis

In this section, we have obtained the Reliability function, hazard function and the reverse hazard rate function of the proposed length biased quasi Shanker distribution.

3.1 Reliability function

The reliability function or the survival function of length biased quasi Shanker distribution is computed as

$$R(x)=1-F_l(x;\theta,\alpha)$$

$$R(x)=1-\frac{1}{(\theta^3+2\theta+6\alpha)}\left(\theta^3\gamma(2,\theta x)+\theta\gamma(3,\theta x)+\alpha\gamma(4,\theta x)\right)$$

3.2 Hazard function

The hazard function is also known as hazard rate defined as the instantaneous failure rate or force of mortality and is given by

$$h(x)=\frac{f_l(x;\theta,\alpha)}{R(x)}$$

$$h(x)=\frac{x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}}{(\theta^3+2\theta+6\alpha)-(\theta^3\gamma(2,\theta x)+\theta\gamma(3,\theta x)+\alpha\gamma(4,\theta x))}$$

3.3 Reverse hazard function

The reverse hazard function of length biased quasi Shanker distribution is given by

$$h_r(x)=\frac{f_l(x;\theta,\alpha)}{F_l(x;\theta,\alpha)}$$

$$h_r(x)=\frac{x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}}{(\theta^3\gamma(2,\theta x)+\theta\gamma(3,\theta x)+\alpha\gamma(4,\theta x))}$$

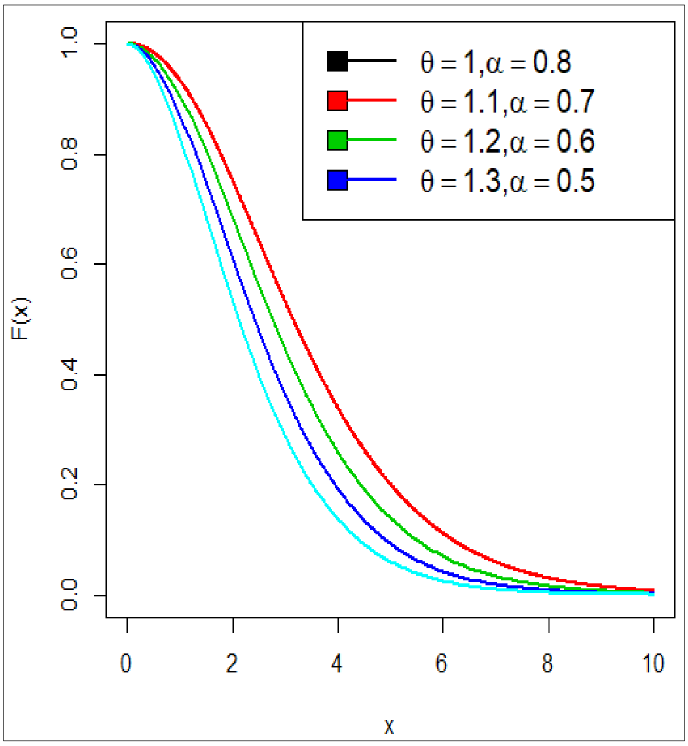

Fig 3: Reliability plot of LBQS distribution

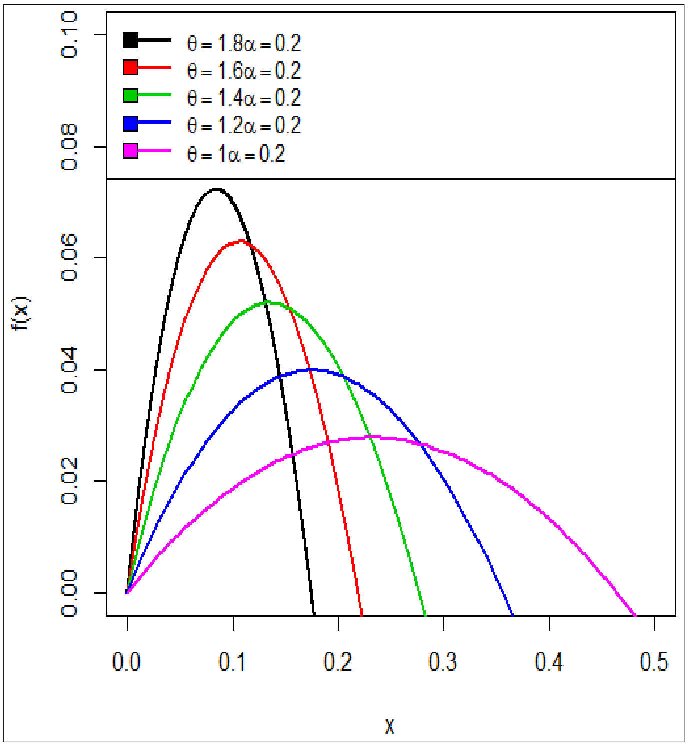

Fig 4: showing harzard of LBQS distribution

4. Moments and Associated Measures: In this section, we will discuss the different statistical properties of length biased quasi Shanker distribution.

4.1 Moments

Let X denotes the random variable of length biased quasi Shanker distribution with parameters θ and α, then the rth order moment $E(X^r)$ of length biased quasi Shanker distribution is obtained as

$$E(X^r)=\mu_r'=\int_0^\infty x^r f_l(x;\theta,\alpha)dx$$

$$E(X^r)=\mu_r'=\int_0^\infty x^r\frac{x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}}{(\theta^3+2\theta+6\alpha)}dx$$

$$E(X^r)=\mu_r'=\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\int_0^\infty x^{r+1}(\theta+x+\alpha x^2)e^{-\theta x}dx$$

$$E(X^r)=\mu_r'=\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\left(\theta\int_0^\infty x^{(r+2)-1}e^{-\theta x}dx+\int_0^\infty x^{(r+3)-1}e^{-\theta x}dx+\alpha\int_0^\infty x^{(r+4)-1}e^{-\theta x}dx\right)$$

After Simplification, we obtain

$$E(X^r) = \mu_r' = \frac{\theta\Gamma(r+2) + \theta^{r+2}\Gamma(r+3) + \alpha\theta^{r+1}\Gamma(r+4)}{\theta^{2r+1}(\theta^3 + 2\theta + 6\alpha)} \qquad (7)$$

Put r = 1, 2, 3 and 4 in equation (7), we get first four moments of length biased weighted quasi Shanker distribution.

$$E(X) = \mu_1' = \frac{2\theta + 6\theta^3 + 24\alpha\theta^2}{\theta^3(\theta^3 + 2\theta + 6\alpha)}$$

$$E(X^2) = \mu_2' = \frac{6\theta + 24\theta^4 + 120\alpha\theta^3}{\theta^5(\theta^3 + 2\theta + 6\alpha)}$$

$$E(X^3) = \mu_3' = \frac{24\theta + 120\theta^5 + 720\alpha\theta^4}{\theta^7(\theta^3 + 2\theta + 6\alpha)}$$

$$E(X^4) = \mu_4' = \frac{120\theta + 720\theta^6 + 5040\alpha\theta^5}{\theta^9(\theta^3 + 2\theta + 6\alpha\theta)}$$

$$\text{Variance } (\mu_2) = \frac{6\theta + 24\theta^4 + 120\alpha\theta^3}{\theta^5(\theta^3 + 2\theta + 6\alpha)} - \left(\frac{2\theta + 6\theta^3 + 24\alpha\theta^2}{\theta^3(\theta^3 + 2\theta + 6\alpha)}\right)^2$$

$$S.D(\sigma) = \sqrt{\left(\frac{6\theta + 24\theta^4 + 120\alpha\theta^3}{\theta^5(\theta^3 + 2\theta + 6\alpha)} - \frac{(2\theta + 6\theta^3 + 24\alpha\theta^2)^2}{(\theta^3(\theta^3 + 2\theta + 6\alpha))^2}\right)}$$

4.2 Harmonic mean

The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals. The harmonic mean for the proposed length biased quasi Shanker distribution is obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \frac{\theta^4(\theta + x + \alpha x^2)e^{-\theta x}}{(\theta^3 + 2\theta + 6\alpha)} dx$$

$$= \frac{\theta^4}{(\theta^3 + 2\theta + 6\alpha)}\left(\theta\int_0^\infty e^{-\theta x}dx + \int_0^\infty xe^{-\theta x}dx + \alpha\int_0^\infty x^2 e^{-\theta x}dx\right)$$

$$= \frac{\theta^4}{(\theta^3 + 2\theta + 6\alpha)}\left(\theta\int_0^\infty e^{-\theta x}x^{(2-2)}dx + \int_0^\infty e^{-\theta x}x^{(2-1)}dx + \alpha\int_0^\infty e^{-\theta x}x^{(3-1)}dx\right)$$

After Simplification, we obtain

$$H.M = \frac{\theta^4}{(\theta^3 + 2\theta + 6\alpha)}\left(\theta\gamma(2,\theta x) + \gamma(2,\theta x) + \alpha\gamma(3,\theta x)\right)$$

4.3 Moment Generating Function

In this sub section we derive the moment generating function and the characteristics function of length biased weighted quasi Shanker distribution. The moment generating function is the expected function of the random variable. We begin with the well-known definition of the moment generating function is given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \ldots\right) f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_l(x;\theta,\alpha)dx$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j'$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \left(\frac{\theta\Gamma(j+2) + \theta^{j+2}\Gamma(j+3) + \alpha\theta^{j+1}\Gamma(j+4)}{\theta^{2j+1}(\theta^3 + 2\theta + 6\alpha)}\right)$$

$$M_X(t) = \frac{1}{\theta(\theta^3 + 2\theta + 6\alpha)} \sum_{j=0}^\infty \frac{t^j}{j!\theta^{2j}} \left(\theta\Gamma(j+2) + \theta^{j+2}\Gamma(j+3) + \alpha\theta^{j+1}\Gamma(j+4)\right)$$

Characteristic function

In probability theory and statistics characteristic function is the function of any real-valued random variable completely defines the probability distribution of a random variable. The characteristic function exists always even when moment generating function does not exists. The characteristic function of length biased quasi Shanker distribution is given by

$$\varphi_X(t) = M_X(it)$$

$$M_X(it) = \frac{1}{\theta(\theta^3 + 2\theta + 6\alpha)} \sum_{j=0}^\infty \frac{(it)^j}{j!\theta^{2j}} \left(\theta\Gamma(j+2) + \theta^{j+2}\Gamma(j+3) + \alpha\theta^{j+1}\Gamma(j+4)\right)$$

5. Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a Continuous distribution with cumulative distribution function $F_X(x)$ and probability density function $f_x(x)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left(F_X(x)\right)^{r-1}\left(1 - F_X(x)\right)^{n-r}, \quad r = 1, 2, 3, \ldots\ldots, n$$

Using equations (5) and (6), the probability density function of r^{th} order statistics of length biased quasi Shanker distribution is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\frac{x\theta^4(\theta + x + \alpha x^2)e^{-\theta x}}{(\theta^3 + 2\theta + 6\alpha)}\right)\left(\frac{1}{(\theta^3 + 2\theta + 6\alpha)}(\theta^3\gamma(2,\theta x) + \theta\gamma(3,\theta x) + \alpha\gamma(4,\theta x))\right)^{r-1}$$
$$\times\left(1 - \frac{1}{(\theta^3 + 2\theta + 6\alpha)}(\theta^3\gamma(2,\theta x) + \theta\gamma(3,\theta x) + \alpha\gamma(4,\theta x))\right)^{n-r}$$

Therefore the probability density function of ist order statistics $X_{(1)}$ of length biased quasi Shanker distribution is given by

$$f_{x(1)}(x) = \frac{nx\theta^4(\theta + x + \alpha x^2)e^{-\theta x}}{(\theta^3 + 2\theta + 6\alpha)}\left(1 - \frac{1}{(\theta^3 + 2\theta + 6\alpha)}(\theta^3\gamma(2,\theta x) + \theta\gamma(3,\theta x) + \alpha\gamma(4,\theta x))\right)^{n-1}$$

and the probability density function of higher order statistics $X_{(n)}$ of length biased quasi Shanker distribution is given by

$$f_{x(n)}(x) = \frac{nx\theta^4(\theta + x + \alpha x^2)e^{-\theta x}}{(\theta^3 + 2\theta + 6\alpha)}\left(\frac{1}{(\theta^3 + 2\theta + 6\alpha)}(\theta^3\gamma(2,\theta x) + \theta\gamma(3,\theta x) + \alpha\gamma(4,\theta x))\right)^{n-1}$$

6. Likelihood Ratio Test

Let $X_1, X_2, \ldots. X_n$ be a random sample of size n drawn from the quasi Shanker distribution or length biased quasi Shanker distribution. We set up the hypothesis for testing.

$H_o : f(x) = f(x;\theta,\alpha)$ against $H_1 : f(x) = f_l(x;\theta,\alpha)$

For testing whether the random sample of size *n* comes from the quasi Shanker distribution or length biased quasi Shanker distribution, the following test statistic is used

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \frac{f_l(x;\theta,\alpha)}{f(x;\theta,\alpha)}$$

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \left(\frac{x_i \theta^4 (\theta^3 + \theta + 2\alpha)}{\theta^3 (\theta^3 + 2\theta + 6\alpha)} \right)$$

$$\Delta = \frac{L_1}{L_o} = \left(\frac{\theta^4 (\theta^3 + \theta + 2\alpha)}{\theta^3 (\theta^3 + 2\theta + 6\alpha)} \right)^n \prod_{i=1}^{n} x_i$$

We should reject the null hypothesis if

$$\Delta = \left(\frac{\theta^4 (\theta^3 + \theta + 2\alpha)}{\theta^3 (\theta^3 + 2\theta + 6\alpha)} \right)^n \prod_{i=1}^{n} x_i > k$$

Equivalently, we also reject the null hypothesis if

$$\Delta^* = \prod_{i=1}^{n} x_i > k \left(\frac{\theta^3 (\theta^3 + 2\theta + 6\alpha)}{\theta^4 (\theta^3 + \theta + 2\alpha)} \right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, \text{Where } k^* = k \left(\frac{\theta^3 (\theta^3 + 2\theta + 6\alpha)}{\theta^4 (\theta^3 + \theta + 2\alpha)} \right)^n$$

Thus for large sample of size *n*, *2log Δ* is distributed as chi-square distribution with one degree of freedom and also p- value is obtained from the chi-square distribution. Also, we reject the null hypothesis, when the probability value is given by

$p(\Delta^* > \lambda^*)$ Where $\lambda^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance and $\prod_{i=1}^{n} x_i$ is the observed value of the statistic Δ^*.

7. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are used not only in economics to study the distribution of income or wealth or income and poverty, but, it is also being used in other fields like reliability, medicine, insurance and demography. The bonferroni and Lorenz curves are one of the important indicators of how wealth is distributed. The bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x f_l(x;\theta,\alpha) dx$$

and $L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x f_l(x;\theta,\alpha) dx$

Where $\mu_1' = E(X) = \frac{(2\theta + 6\theta^3 + 24\alpha\theta^2)}{\theta^3 (\theta^3 + 2\theta + 6\alpha)}$ and $q = F^{-1}(p)$

$$B(p) = \frac{\theta^3 (\theta^3 + 2\theta + 6\alpha)}{p(2\theta + 6\theta^3 + 24\alpha\theta^2)} \int_0^q \frac{x^2 \theta^4 (\theta + x + \alpha x^2) e^{-\theta x}}{(\theta^3 + 2\theta + 6\alpha)} dx$$

$$B(p)=\frac{\theta^7}{p(2\theta+6\theta^3+24\alpha\theta^2)}\int_0^q x^2(\theta+x+\alpha x^2)e^{-\theta x}dx$$

$$B(p)=\frac{\theta^7}{p(2\theta+6\theta^3+24\alpha\theta^2)}\left(\theta\int_0^q e^{-\theta x}x^{(3-1)}dx+\int_0^q e^{-\theta x}x^{(4-1)}dx+\alpha\int_0^q e^{-\theta x}x^{(5-1)}dx\right)$$

After simplification, we obtain

$$B(p)=\frac{\theta^7}{p(2\theta+6\theta^3+24\alpha\theta^2)}\left(\theta\gamma(3,\theta q)+\gamma(4,\theta q)+\alpha\gamma(5,\theta q)\right)$$

$$L(p)=pB(p)=\frac{\theta^7}{(2\theta+6\theta^3+24\alpha\theta^2)}\left(\theta\gamma(3,\theta q)+\gamma(4,\theta q)+\alpha\gamma(5,\theta q)\right)$$

8. Entropies

The idea of entropy provides a mathematical way to encode the intuitive notion of which processes are impossible, even though they would not violate the fundamental law. The concept of entropies is important in different areas such as probability and statistics, physics, communication theory and economics. Entropies quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable X is a measure of variation of the uncertainty.

8.1 Renyi Entropy

The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$e(\beta)=\frac{1}{1-\beta}\log\left(\int f_l^{\beta}(x)dx\right)$$

Where, $\beta>0$ and $\beta\neq 1$

$$e(\beta)=\frac{1}{1-\beta}\log\int_0^\infty\left(\frac{x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}}{(\theta^3+2\theta+6\alpha)}\right)^{\beta}dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\right)^{\beta}\int_0^\infty x^{\beta}e^{-\theta\beta x}\left(\theta+x+\alpha x^2\right)^{\beta}dx\right)$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\right)^{\beta}\left(\theta+x+\alpha x^2\right)^{\beta}\int_0^\infty e^{-\theta\beta x}x^{(\beta+1)-1}dx\right)$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\right)^{\beta}\left(\theta+x+\alpha x^2\right)^{\beta}\frac{\Gamma(\beta+1)}{(\theta\beta)^{\beta+1}}\right)$$

8.2 Tsallis Entropy

Tsallis in 1988 introduced an entropic expression characterized by an index q which leads to a non-extensive statistics. Tsallis entropy is the basis of the so called non-extensive statistical mechanics. A generalization of Boltzmann-Gibbs (B.G) statistical properties initiated by Tsallis has focused a great deal to attention.

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^\infty f_l^{\lambda}(x)dx\right)$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^\infty\left(\frac{x\theta^4(\theta+x+\alpha x^2)e^{-\theta x}}{(\theta^3+2\theta+6\alpha)}\right)^{\lambda}dx\right)$$

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\left(\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\right)^\lambda \int_0^\infty x^\lambda e^{-\lambda\theta x}(\theta+x+\alpha x^2)^\lambda dx\right)$$

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\left(\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\right)^\lambda (\theta+x+\alpha x^2)^\lambda \int_0^\infty e^{-\lambda\theta x} x^{(\lambda+1)-1} dx\right)$$

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\left(\frac{\theta^4}{(\theta^3+2\theta+6\alpha)}\right)^\lambda (\theta+x+\alpha x^2)^\lambda \frac{\Gamma(\lambda+1)}{(\lambda\theta)^{\lambda+1}}\right)$$

9. Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the maximum likelihood estimation for estimating the parameters of the proposed distribution and also the Fisher's Information matrix has been discussed. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the length biased quasi Shanker distribution, then the corresponding likelihood function is given by

$$L(x) = \prod_{i=1}^n f_l(x;\theta,\alpha)$$

$$L(x) = \prod_{i=1}^n \left(\frac{x_i\theta^4(\theta+x_i+\alpha x_i^2)e^{-\theta x_i}}{(\theta^3+2\theta+6\alpha)}\right)$$

$$L(x) = \frac{\theta^{4n}}{(\theta^3+2\theta+6\alpha)^n}\prod_{i=1}^n\left(x_i(\theta+x_i+\alpha x_i^2)e^{-\theta x_i}\right)$$

The log likelihood function is given by

$$\log L(x) = 4n\log\theta - n\log(\theta^3+2\theta+6\alpha) + \sum_{i=1}^n \log x_i + \sum_{i=1}^n \log(\theta+x_i+\alpha x_i^2) - \theta\sum_{i=1}^n x_i$$

The maximum likelihood estimate of the parameters θ and α can be obtained by differentiating equation (9) with respect to parameters θ and α. We obtain the system of normal equations as

$$\frac{\partial\log L}{\partial\theta} = \frac{4n}{\theta} - n\left(\frac{3\theta^2+2}{(\theta^3+2\theta+6\alpha)}\right) + \sum_{i=1}^n\left(\frac{1}{(\theta+x_i+\alpha x_i^2)}\right) - \sum_{i=1}^n x_i = 0$$

$$\frac{\partial\log L}{\partial\alpha} = -n\left(\frac{6}{(\theta^3+2\theta+6\alpha)}\right) + \frac{\sum_{i=1}^n x_i^2}{(\theta+x_i+\alpha x_i^2)} = 0$$

Because of the complicated form of the likelihood equations, algebraically it is very difficult to solve the system of non-linear equations. Therefore we use R and wolfram mathematics for estimating the required parameters of the proposed distribution.

To obtain confidence interval we use the asymptotic normality results. We have that if $\hat{\lambda} = (\hat{\theta},\hat{\alpha})$ denotes the MLE of $\lambda = (\theta,\alpha)$ .we can state the results as follows

$$\sqrt{n}(\hat{\lambda}-\lambda) \rightarrow N_2(0, I^{-1}(\lambda))$$

Where $I^{-1}(\lambda)$ is the Fisher information matrix

The elements of 2×2 Fisher's Information matrix is given below

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) \end{pmatrix}$$

Where

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{4n}{\theta^2} - n\left(\frac{(\theta^3+2\theta+6\alpha)6\theta - (3\theta^2+2)(3\theta^2+2)}{(\theta^3+2\theta+6\alpha)^2}\right) - \sum_{i=1}^{n}\left(\frac{1}{(\theta+x_i+\alpha x_i^2)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\left(\frac{36}{(\theta^3+2\theta+6\alpha)^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i^4)}{(\theta+x_i+\alpha x_i^2)^2}\right)$$

Also,

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) = -n\left(\frac{6(3\theta^2+2)}{(\theta^3+2\theta+6\alpha)^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i^2)}{(\theta+x_i+\alpha x_i^2)^2}\right)$$

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for θ and α

10. Data Evaluation

In this section, here we investigate the two real-life data sets for fitting the length biased quasi shanker distribution in order to show that the proposed length biased quasi Shanker distribution fits better than the quasi shanker, Shanker, exponential and lindley distributions. The following two real data sets are given below as

Data Set 1: The following data set represent the data set of 40 patients suffering from blood cancer (leukaemia) from one of ministry of health hospitals in saudi Arabia (see Abouammah *et al*.). The data set is provided below in table 1.

Table 1: Data regarding the blood cancer (leukaemia) patients (n=40) from ministry of health hospitals in Saudi Arabia

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036
2.162	2.211	2.37	2.532	2.693	2.805	2.91	2.912	3.192
3.263	3.348	3.348	3.427	3.499	3.534	3.767	3.751	3.858
3.986	4.049	4.244	4.323	4.381	4.392	4.397	4.647	4.753
4.929	4.973	5.074	5.381					

Data set 2: The second data set is reported by Xu *et al*. (2003).The following data represents the time to the failure of (103h) of turbo charger one type of engine. The data is represented below in table 2.

Table 2: Data regarding the time to failure of turbocharger (n=40) studied by Xu *et al*. (2003)

1.6	3.5	4.8	5.4	6.0	6.5	7	7.3	7.7	8
8.4	2	3.9	5	5.6	6.1	6.5	7.1	7.3	7.8
8.1	8.4	2.6	4.5	5.1	5.8	6.3	6.7	7.3	7.7
7.9	8.3	8.5	3	4.6	5.3	6	8.7	8.8	9

The unknown parameters of the model along with criterion values are estimated by using the R Software. In order to compare the length biased quasi Shanker distribution with quasi Shanker, Shanker, exponential and Lindley distributions, we employ the criterion values like *AIC* (Akaike information criterion), *AICC* (corrected Akaike information criterion) and *BIC* (Bayesian information criterion). The better distribution corresponds to minimum values of *AIC, AICC, BIC* and *-2logL*. The generic formulas for *AIC, AICC* and *BIC* are given as

$\text{AIC} = 2k - 2\log L$ $\quad$ $\text{AICC} = AIC + \frac{2k(k+1)}{n-k-1}$ $\quad$ and $\quad$ $\text{BIC} = k\log n - 2\log L$

Where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model.

Table 3: Shows parameter estimates, corresponding S. errors, Criterion values and Performance of the fitted distribution

Data sets	Distribution	MLE	S.E	-2logL	AIC	BIC	AICC
1	Length Biased Quasi Shanker	$\hat{\alpha} = 2.69563$ $\hat{\theta} = 1.18535$	$\hat{\alpha} = 3.32014$ $\hat{\theta} = 0.12113$	144.8703	148.8703	152.248	149.19462
	Quasi Shanker	$\hat{\alpha} = 21.2671$ $\hat{\theta} = 0.93676$	$\hat{\alpha} = 58.7944$ $\hat{\theta} = 0.09723$	147.3439	151.3439	154.7216	151.66822
	Shanker	$\hat{\theta} = 0.54874$	$\hat{\theta} = 0.0572$	157.8358	159.8358	161.5247	159.9410
	Exponential	$\hat{\theta} = 0.31839887$	$\hat{\theta} = 0.05034278$	171.5563	173.5563	175.2452	173.6615
	Lindley	$\hat{\theta} = 0.52692$	$\hat{\theta} = 0.06074$	160.5012	162.5012	164.19	162.6064
2	Length Biased Quasi Shanker	$\hat{\alpha} = 1.37880$ $\hat{\theta} = 6.39708$	$\hat{\alpha} = 1.67778$ $\hat{\theta} = 5.05740$	182.2407	186.2407	189.6185	186.56502
	Quasi Shanker	$\hat{\alpha} = 1.328420$ $\hat{\theta} = 4.797644$	$\hat{\alpha} = 1.677870$ $\hat{\theta} = 4.379816$	189.0107	193.0107	196.3884	193.33502
	Shanker	$\hat{\theta} = 0.30167$	$\hat{\theta} = 0.032586$	203.8728	205.8728	207.5617	205.9780
	Exponential	$\hat{\theta} = 0.15993$	$\hat{\theta} = 0.02528746$	226.6385	228.6385	230.3274	228.7437
	Lindley	$\hat{\theta} = 0.28448$	$\hat{\theta} = 0.03219$	208.5708	210.5708	212.2597	210.6760

From table 3, it has been observed from the results that the length biased quasi Shanker distribution have the lesser *AIC, AICC, BIC* and *-2*logL values as compared to the quasi Shanker, Shanker, exponential and lindley distributions, which clearly indicated that the length biased quasi Shanker distribution fits better than the quasi Shanker, Shanker, exponential and lindley distributions. Hence we can conclude that the length biased quasi Shanker distribution leads to a better fit than the quasi Shanker, Shanker, exponential and lindley distributions.

11. Conclusion

This manuscript deals with the new modification of quasi Shanker distribution known as length biased quasi shanker distribution. The subject distribution is generated by using the length biased technique and taking the two parameter quasi Shanker distribution as the base distribution. The different statistical properties of the executed distribution are derived and discussed in detail. The parameters of the proposed distribution are also estimated by using the method of maximum likelihood estimation and also the Fisher's information matrix have also been discussed. The usefulness and supremacy of the proposed distribution has also been investigated by fitting the two real lifetime data sets and then it is found from the results of two data sets that the length biased quasi Shanker distribution fits quite satisfactory over quasi shanker, Shanker, exponential and lindley distributions. Hence, we can conclude that the length biased quasi Shanker distribution fits better than the quasi Shanker, Shanker, exponential and lindley distributions.

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