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Twisting of a rigid cylinder in an infinite elastic medium of dielectric material

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Abstract
The present problem is investigated in context of finite deformation theory. In this paper, the author studied the effect of polarization on twisting of a rigid cylinder in an infinite elastic medium.

Keywords: Finite deformation, dielectric material, polarization, isotropic, electrostatic

1. Introduction

A dielectric material is a substance that is a poor conductor of electricity, but an efficient supporter of electrostatic fields. If the flow of current between opposite electric charge poles is kept to a minimum while the electrostatic lines of flux are not impeded or interrupted, an electrostatic field can store energy. Due to this property dielectric materials are vital component of capacitors, electronic devices which can store charge. Eringen [1] gave modified form of Toupin's theory [17] of elastic dielectric and obtained specific forms of the basic field equations, the boundary conditions and the constitutive equations that must satisfies the stress, electrical and polarization fields by using a variational principle in electro elastostatics. Recently authors [10-12] and Singh [15] have investigated some basic problems of practical interest for circular cylinders composed of isotropic hyperelastic incompressible materials by using this theory. The problem of circular shearing (azimuthal shear) of a compressible hyperelastic cylinder has been studied by Ertepinar [3], Haughton [6], Jiang and Ogden [9], Polgnone and Horgan [14], Simmonds ans Warne [16], Tao *et al*. [18], Wneman and Waldron [19]. Shear problems in circular cylinders for incompressible materials with limiting chain extensibility have been investigated by Horgan and Saccomandi [7-8]. The present research is related to examine the effect of polarization to the problem of azimuthal shear of a hollow circular dielectric cylindrical tube. The inner surface of the tube is bonded to a rigid cylinder and uniformly distributed azimuthal shear traction is applied to the outer surface of the tube with zero radial traction maintained at the same surface. The formulation of the problem is based on the theory of finite elastic deformations formulated by Eringen [2], Green and Zerna [4], Green and Adkin [5], ogden [13].

2. Fundamental equations

The basic equations of an incompressible, homogeneous, isotropic, hyperelastic dielectric can be classified in the following three groups:

(a). Field equations

(2.1) $t^k_{l;k} + \rho f_l = 0,$

(2.2) ${}_L E_k - \phi_{,k} = 0,$

(2.3) $\varepsilon_0 \nabla^2 \phi - div\vec{P} = -q_f, \quad in \quad V_d,$

where t^k_l is the Cauchy stress tensor, ρ is the volume density, f_l is the body force per unit mass, ${}_L E_k$ is the local electrostatic field, ε_0 is the material constant, ϕ is the electrostatics potential, $\vec{P}$ is the polarization vector, q_f is the volume free charge, V_d is the volume that the dielectric occupies. Semicolon and comma indicate the covariant and partial derivatives respectively.

(b). Boundary conditions

(2.4) $\left\| t_l^k \right\| n_k = 0,$

(2.5) $\left\| \varepsilon_0 \phi_{,}^k - \vec{P}^k \right\| n_k + \omega_f = 0, \quad on \quad S_d,$

where n_k is the exterior normal to S_d, S_d is the surface enclosing the dielectric volume, ω_f is the free surface charge, the double bracket stands for discontinuity across the surface. The Cauchy stress tensor t_l^k is defined as

(2.6) $t_l^k \equiv {}_L t_l^k + {}_M t_l^k,$

(2.7) ${}_M t_l^k \equiv \varepsilon_0 (\phi_{,}^k \phi_{,l} - 1/2 \phi_{,}^m \phi_{,m} \delta_l^k),$

where ${}_M t_l^k$ is the Maxwell stress tensor.

(c). Constitutive equations

(2.8) $${}_L t_l^k = -p\delta_l^k + 2[{}^{-1}c_l^k (\frac{\partial \Sigma}{\partial I_1} + I_1 \frac{\partial \Sigma}{\partial I_2}) - {}^{-2}c_l^k \frac{\partial \Sigma}{\partial I_2} + {}^{-1}c_m^k P^m P_l \frac{\partial \Sigma}{\partial I_4}$$

$$+ {}^{-2}c_m^k P^m P_l \frac{\partial \Sigma}{\partial I_5} + ({}^{-1}c_m^k)({}^{-1}c_l^n) P^m P_n \frac{\partial \Sigma}{\partial I_5}],$$

(2.9) $${}_L E^k = 2[{}^{-1}c_l^k \frac{\partial \Sigma}{\partial I_4} + {}^{-2}c_l^k \frac{\partial \Sigma}{\partial I_5} + \delta_l^k \frac{\partial \Sigma}{\partial I_6}] P^l,$$

where p is the arbitrary hydrostatic pressure, δ_l^k is a Kronecker delta, $\Sigma = \Sigma(I_1, I_2, I_4, I_5, I_6)$ and $I's$ are the invariants based on Finger's strain measure ${}^{-1}c$ and polarization $\vec{P}$. These are given by

(2.10) $$I = I_1 = \delta_l^k \, {}^{-1}c_l^k, \; II = I_2 = \frac{1}{2}\delta_{ln}^{km} \, {}^{-1}c_k^l \, {}^{-1}c_m^n, \; III = I_3 = \frac{1}{6}\delta_{lnq}^{kmp} \, {}^{-1}c_k^l \, {}^{-1}c_m^n \, {}^{-1}c_p^q,$$

$$I_4 = {}^{-1}c_l^k P^l P_k, \; I_5 = {}^{-1}c_m^k \, {}^{-1}c_l^m P^l P_k, \; I_6 = P^2.$$

The deformation tensors c_l^k and ${}^{-1}c_l^k$ are given by

$${}^{-1}c_l^k = g_{ml} G^{KM} x_{,K}^k x_{,M}^m,$$

(2.11) $$c_l^k = g^{km} G_{ML} X_{,m}^M X_{,l}^L.$$

3. Formulation of the problem

For the incompressible tube, with inner surface bonded to a rigid cylinder and a uniformly distributed azimuthal shear traction applied to the outer surface, the deformation is that of pure azimuthal shear (no radial deformation) described by

(3.1) $r = R, \quad \theta = \Theta + g(R), \quad z = Z,$

where the material and spatial cylindrical polar coordinates are denoted by (R, Θ, Z) and (r, θ, z) respectively, with $a \le R \le b$.

Using (2.11), we can find the deformation tensors as

(3.2) $\|c_{kl}\| = \begin{bmatrix} 1+R^2g'^2 & -Rg' & 0 \\ -Rg' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \|{}^{-1}c_{kl}\| = \begin{bmatrix} 1 & Rg' & 0 \\ Rg' & 1+R^2g'^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

With the help of (2.10) and (3.2), we calculate the principal invariants

(3.3) $I_1 = I_2 = 3 + R^2g'^2, \quad I_3 = 1, \quad I_4 = P^2, \quad I_5 = (1+R^2g'^2)P^2, \quad I_6 = P^2.$

4. Electrostatic and Maxwell fields

To determine the electrostatic field, we solve (2.3) with the boundary conditions (2.4) and (2.5). We assume that electric field and polarization field have single component i.e., $\vec{E} = [E(R),0,0], \vec{P} = [P(R),0,0].$
As a final result (see Eringen [2]), we find

$$\phi = \alpha_1, \qquad for \quad 0 < R < a$$

(4.1) $\varepsilon_0\phi = -a\omega_f \log R + \int^R P(R)dR + \beta_2, \qquad for \quad a < R < b$

$$\varepsilon_0\phi = -a\omega_f \log R, \qquad for \quad R > b.$$

The unknown constants α_1 and β_2 are immaterial for electric and stress field. Using $(3.1), (4.1)_2, (2.7)$, we obtain

(4.2) ${}_M t_1^1 = -{}_M t_2^2 = -{}_M t_3^3 = \frac{1}{2\varepsilon_0}(\frac{a\omega_f}{R} - P)^2.$

The equation ${}_M\vec{E} = -grad\,\phi$ gives the Maxwell electric field as follows

(4.3) ${}_M E^1 = \frac{1}{\varepsilon_0}(\frac{a\omega_f}{R} - P)^2, \quad {}_M E^2 = 0, \quad {}_M E^3 = 0.$

5. Local electric and stress fields. By using (3.2), we obtain the local electric field from (2.9) as

$${}_L E^1 = 2[\frac{\partial\Sigma}{\partial I_4} + (1+R^2g'^2)\frac{\partial\Sigma}{\partial I_5} + \frac{\partial\Sigma}{\partial I_6}]P,$$

(5.1) ${}_L E^2 = 2[Rg'\frac{\partial\Sigma}{\partial I_4} + (2Rg' + R^3g'^3)\frac{\partial\Sigma}{\partial I_5}]P,$

$${}_L E^3 = 0.$$

The local stress tensor from (2.8) for incompressible dielectric is obtained as

$${}_L t_1^1 = -p + 2[(\frac{\partial\Sigma}{\partial I_1} + 2\frac{\partial\Sigma}{\partial I_2}) + \frac{\partial\Sigma}{\partial I_4} + (2+R^2g'^2)\frac{\partial\Sigma}{\partial I_5}P^2],$$

(5.2) ${}_L t_2^2 = -p + 2[(1+R^2g'^2)\frac{\partial\Sigma}{\partial I_1} + (2+R^2g'^2)\frac{\partial\Sigma}{\partial I_2} + R^2g'^2P^2\frac{\partial\Sigma}{\partial I_5}],$

$${}_L t_3^3 = -p + 2[\frac{\partial\Sigma}{\partial I_1} + (2+R^2g'^2)\frac{\partial\Sigma}{\partial I_2}],$$

$${}_L t_2^1 = 2[\frac{\partial\Sigma}{\partial I_1} + \frac{\partial\Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial\Sigma}{\partial I_5}],$$

$${}_L t_3^1 = 0, \quad {}_L t_3^2 = 0.$$

By using (2.6), (5.1) and (5.2), the components of Cauchy stress tensor can be written in the form

$$t_1^1 = p + 2[(\frac{\partial\Sigma}{\partial I_1} + 2\frac{\partial\Sigma}{\partial I_2}) + \frac{\partial\Sigma}{\partial I_4} + (2 + R^2 g'^2)\frac{\partial\Sigma}{\partial I_5} P^2] - \frac{1}{2\varepsilon_0}(\frac{a\omega_f}{R} - P)^2,$$

$$t_2^2 = -p + 2[(1 + R^2 g'^2)\frac{\partial\Sigma}{\partial I_1} + (2 + R^2 g'^2)\frac{\partial\Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial\Sigma}{\partial I_5}] + \frac{1}{2\varepsilon_0}(\frac{a\omega_f}{R} - P)^2,$$

(5.3) $$t_3^3 = -p + 2[\frac{\partial\Sigma}{\partial I_1} + (2 + R^2 g'^2)\frac{\partial\Sigma}{\partial I_2}] + \frac{1}{2\varepsilon_0}(\frac{a\omega_f}{R} - P)^2,$$

$${}_L t_2^1 = 2[\frac{\partial\Sigma}{\partial I_1} + \frac{\partial\Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial\Sigma}{\partial I_5}],$$

$${}_L t_3^1 = 0,$$

$${}_L t_3^2 = 0.$$

The equations of force equilibrium with the vanishing body force are

(5.4) $$\frac{\partial t_1^1}{\partial R} + \frac{1}{R}(t_1^1 - t_2^2) = 0,$$

(5.5) $$\frac{\partial t_2^1}{\partial R} + 2\frac{t_2^1}{R} = 0.$$

From (5.5), we have

(5.6) $$\frac{d}{dR}(R^2 t_2^1) = 0.$$

On integration (5.6), we find that

(5.7) $$t_2^1 = \frac{b^2}{R^2} T_0,$$

where T_0 is the prescribed azimuthal shear stress on the outer boundary. On using (5.3), we obtain a first order differential equation for $g(R)$, namely

(5.8) $$2Rg'[\frac{\partial\Sigma}{\partial I_1} + \frac{\partial\Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial\Sigma}{\partial I_5}] = \frac{b^2}{R^2} T_0,$$

also

(5.9) $$\frac{\partial t_1^1}{\partial R} - \frac{2}{R}[R^2 g'^2 (\frac{\partial\Sigma}{\partial I_1} + \frac{\partial\Sigma}{\partial I_2}) - (\frac{\partial\Sigma}{\partial I_4} + 2\frac{\partial\Sigma}{\partial I_5})P^2 - \frac{1}{2\varepsilon_0}(\frac{a\omega_f}{R} - P)^2] = 0.$$

Obtain $g(R)$ from (5.8), subject to the boundary condition

(5.10) $g(a) = 0.$

Integrating (5.10) from R to b and use of the boundary condition

(5.11) $t_1^1(b) = 0.$

Yields

(5.12) $t_1^1(R) = \int_R^b \frac{-2}{s}[s^2 g'^2 (\frac{\partial\Sigma}{\partial I_1} + \frac{\partial\Sigma}{\partial I_2}) - (\frac{\partial\Sigma}{\partial I_4} + 2\frac{\partial\Sigma}{\partial I_5})P^2 - \frac{1}{2\varepsilon_0}(\frac{a\omega_f}{s} - P)^2]ds.$

6. Twisting of a rigid cylinder in an infinite elastic medium. Another interesting set of boundary conditions, for a hollow tube surrounded by a rigid casing, is

(6.1) $g(a) = g_0, \quad g(b) = 0,$

In this case, integration of (8.2) for dielectric material gives

(6.2) $g(\overline{R}) = \frac{g_0(\overline{R}^2 - \eta^2)}{\overline{R}^2(\eta^2 - 1)}.$

And the normal stresses are

$$t_1^1 = -2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2 - 1)^2}(\frac{\eta}{\overline{R}})^4 + \frac{1}{2}\frac{1}{\varepsilon_0}(1 + K_1)^2 \omega_f^2 (\frac{1}{\overline{R}})^2,$$

(6.3) $t_2^2 = 2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2 - 1)^2}[3(\frac{\eta}{\overline{R}})^4] + \frac{3}{2}\frac{1}{\varepsilon_0}(1 + K_1)^2 \omega_f^2 (\frac{1}{\overline{R}})^2,$

$$t_3^3 = -2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2 - 1)^2}(\frac{b}{\overline{R}})^4 + \frac{3}{2}\frac{1}{\varepsilon_0}(1 + K_1)^2 \omega_f^2 (\frac{1}{\overline{R}})^2.$$

These solutions may be simplified on considering the limit as $\eta \to \infty$ i.e. the boundary-value problem corresponding to the twisting of a rigid cylinder of radius bonded to an infinite elastic medium. In this case, we have from (9.2)

(6.4) $g(\overline{R}) = \frac{g_0}{\overline{R}^2}.$

From (6.3) and (6.4), we get

$$t_1^1 = -2(\alpha_1 + \alpha_2)g^2(\overline{R}) + \frac{1}{2}\frac{1}{\varepsilon_0}(1 + K_1)^2 \omega_f^2 (\frac{1}{\overline{R}})^2,$$

(6.5) $t_2^2 = 6(\alpha_1 + \alpha_2)g^2(\overline{R}) + \frac{3}{2}\frac{1}{\varepsilon_0}(1 + K_1)^2 \omega_f^2 (\frac{1}{\overline{R}})^2,$

$$t_3^3 = -2(\alpha_1 + \alpha_2)g^2(\overline{R}) + \frac{3}{2}\frac{1}{\varepsilon_0}(1 + K_1)^2 \omega_f^2 (\frac{1}{\overline{R}})^2.$$

From (6.5), shows the dependence of normal stresses on polarization. In the absence of polarization, we obtain results similar to the results obtained by Horgan and Saccomandi [8] for isotropic hyperelastic materials with limiting chain extensibility also from (6.5) axial and radial stresses are equal in the absence of radial electric field

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