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New approach to some results related to mixed norm double sequence spaces

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Abstract
In this paper, the mixed norm double sequence spaces ${}^{2}\ell^{p,q}$ for $1 \le p, q \le \infty$ are the subject of our research; we establish conditions for an operator T_λ to be compact, where T_A is given by a diagonal matrix. This will be achieved by applying the Hausdroff measure of non-compactness and the theory of BK spaces. This problem was treated and solved in, but in a different way, without the application of the theory of infinite matrices and BK spaces. Here, we will present a new approach to the problem. Some of our results are known and others are new.

Keywords: Mixed norm double sequence spaces, measures of non-compactness, BK spaces

Introduction
As usual, let ω denote the set of all complex sequences $x=(x_{mn})_{m,n=0}^{\infty}$, ℓ_∞^2 the set of all bounded sequences in ω, and $\ell_p^2 = \left\{ x \in \omega \,\middle|\, \sum\sum_{2 \le m+n \le k} |x_{mn}|^p < \infty \right\}$ for $0 < p < \infty$.

The spaces ${}^{2}\ell^{p,q}$ were introduced by Kellogg in [7] and further studied by many authors [1-6]; they are defined for $1 \le p, q \le \infty$ by

$$ {}^{2}\ell^{p,q} = \left\{ x \in \omega \,\middle|\, \sum\sum_{2 \le m+n \le k} \left\| \left(\sum\sum_{2 \le i+j \le I(m+n)} |x_{mn}|^p \right)^{\frac{q}{p}} < \infty \right. \right\} \quad (1) $$

where $I(0) = \{0\}$ and $I(m, n) = \left\{ i, j \in N \,\middle|\, 2^{i-1,\, j-1} \le m+n < 2^{i+j} \right\}$ for $m > 0$. They are Banach spaces equipped with the norms

$$ \|x\|_{p,q} = \left\{ \sum\sum_{2 \le m+n \le k} \left(\sum\sum_{2 \le i+j \le I(m+n)} |x_{mn}|^p \right)^{q/p} \right\}^{1/q} \quad (2) $$

and referred to as mixed norm spaces.
Let us remark that we could generalize our research by replacing dyadic blocks by arbitrary blocks, similarly as in [1]. In that case we would have $I_{(i,j)} = \{ m, n \in N \mid k(i+j) \le m+n \le k(i+1,\ j+1)-1 \}$ for i, j= 0, 1, 2, … where $(k(i,j))_{i,j=0}^{\infty}$ is a strictly increasing sequence of integers with k(0) = 0.

If $p = \infty$ or $q = \infty$, the corresponding sum should be replaced by the supremum, that is,
for $p = \infty$ and $1 \le q < \infty$,

$$^2\ell^{\infty,q} = \left\{ x \in \omega \,\middle|\, \sum\sum_{2\le i+j\le k} \left(\sup_{m,n\in I(i,j)} |x_{mn}| \right)^q < \infty \right\} \tag{3}$$

with the norm

$$\|x\|_{\infty,q} = \left(\sum\sum_{2\le i+j\le k} \left(\sup_{m,n\in I(i,j)} |x_{mn}|^q \right)\right)^{1/q} \tag{4}$$

for $1 \le p < \infty$ and $q = \infty$,

$$^2\ell^{p,\infty} = \left\{ x \in \omega \,\middle|\, \sup_{i,j} \left(\sum\sum_{2\le m+n\le I(i,j)} |x_{mn}|^p \right)^{1/p} < \infty \right\} \tag{5}$$

with the norm

$$\|x\|_{p,\infty} = \sup_{i,j} \left(\sum\sum_{2\le m+n\le I(i,j)} |x_{mn}|^p \right)^{1/p} \tag{6}$$

for $p = q = \infty$,

$$^2\ell^{\infty,\infty} = \left\{ x \in \omega \,\middle|\, \sup_{i,j} \left(\sup_{2\le m,n\le I(i,j)} |x_{mn}| \right) < \infty \right\} \tag{7}$$

with the norm

$$\|x\|_{\infty,\infty} = \sup_{i,j} \left(\sum\sum_{2\le m,n\le I(i,j)} |x_{mn}| \right) \tag{8}$$

Note that $^2\ell^{p,p} = {}^2\ell^p$ for $1 \le p \le \infty$.

For any two subset E and F of ω, the set of multipliers from E to F, is defined as
M(E, F) =

$$\left\{ \lambda = (\lambda_{m'n'})_{m',n'=0}^{\infty} \in \omega \,\middle|\, \lambda_x = (\lambda_{m'n'} x_{m'n'})_{m',n'=0}^{\infty} \in F \text{ for each } x = (x_{m'n'})_{m',n'=0}^{\infty} \text{ in } E \right\}.$$

The following results are of interest for the characterizations of the multipliers $M\left({}^2\ell^{r,s}, {}^2\ell^{u,v}\right)$.

Theorem

[7, Theorem 1] Let $1 \le r, s, u, v \le \infty$, and define p and q by

$\frac{1}{p} = \frac{1}{u} - \frac{1}{r}$ if r > u, $\quad p = \infty$ if $r \le u$,

$\frac{1}{q} = \frac{1}{v} - \frac{1}{s}$ if s > v, $\quad q = \infty$ if $s \le v$.

Then $M\left({}^{2}\ell^{r,s},\ {}^{2}\ell^{u,v}\right) = {}^{2}\ell^{p,q}$.

Further, for $\lambda \in M\left({}^{2}\ell^{r,s},\ {}^{2}\ell^{u,v}\right)$, let us consider operator $T_\lambda : {}^{2}\ell^{r,s} \to {}^{2}\ell^{u,v}$ defined by

$$T_\lambda(x) = \lambda_x = (\lambda_{mn} x_{mn})_{m,n=0}^{\infty} \left(x \in {}^{2}\ell^{r,s}\right).$$

Kellogg [7] proved that T_λ defined in such a way is a bounded linear operator with norm $\|T_\lambda\| = \|\lambda\|_{p,q}$, where r, s, u, v, p and q satisfy the conditions stated in the previous theorem [5]. Actually, $T_\lambda : {}^{2}\ell^{r,s} \to {}^{2}\ell^{u,v}$ is a bounded linear operator if and only if $\lambda \in {}^{2}\ell^{p,q}$.

In [5, 6], the authors studied the Hausdorff measure of non-compactness of the operator T_λ depending on different cases and found the exact measure or gave estimates for it. They did, however, not make use of any relation between matrix transformation between sequence spaces and bounded linear operators.

Our idea is to give a new approach to the same problem, by using the theory of BK spaces and matrix transformations.

We use the following standard notations.

We write B(X, Y) for the set of all bounded linear operators between the normed spaces X and Y. If X and Y are any subsets of ω, then (X, Y) denotes the set of all infinite matrices $A = \left(a_{m,n,m',n'}\right)_{m,n,m',n'=0}^{\infty}$ that map X into Y, that is, $A \in (X, Y)$ if and only if the series $A_{mn}(x) = \sum\sum_{2 \le m'+n' \le k} a_{mn\,m'n'}\, x_{m'n'}$ converges for all m, n = 0, 1, … and all $x \in X$, and $Ax = \left(A_{mn}x\right)_{m,n=0}^{\infty} \in Y$ for all $x \in X$. We write $A_{m,n} = \left(a_{m,n,m',n'}\right)_{m',n'=0}^{\infty}$ for the sequence in the n-th row of the matrix A.

A BK-space is a Banach sequence space X with continuous coordinates P_{mn} (n = 0, 1, …) where $P_{mn}(x) = x_{mn}$ for each sequence $x = \left(x_{m',n'}\right)_{m',n'=0}^{\infty} \in X$. By ϕ we denote the set of all finite sequences. A BK-space $X \supset \phi$ is said to have AK if $x^{[i,j]} = \sum\sum_{2 \le m'+n' \le i+j} x_{m'n'}\, e^{(m'+n')} \to x\ (i + j \to \infty)$ for every sequence $x = \left(x_{m',n'}\right)_{m',n'=0}^{\infty} \in X$.

It is known [4, Example 3.4(a)] that the space ${}^{2}\ell^{p,q}$ is a BK-space with AK for $1 \le p \le \infty$, $1 \le q < \infty$. Also, the space ${}^{2}\ell^{\infty,q}$ is a BK-space for $1 \le q \le \infty$.

We need the following important result.

Proposition: Let X and Y be BK-spaces.

Then we have $(X, Y) \subset B(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in B(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$ [10, Theorem 4.2..8].

If X has AK then we have $B(X, Y) \subset (X, Y)$ that is, every operator $L \in B(X, Y)$ is given by a matrix $A \in (X, Y)$, where $Ax = L(x)$ for all $x \in X$[3, Theorem 1.9].

Hence, we can consider the infinite matrix $A = A(\lambda) = \left(a_{m,n,m',n'}\right)_{m,n,m',n'=0}^{\infty}$ associated to the operator T_λ such that $T_\lambda x = Ax$ for all $x \in {}^{2}\ell^{r,s}$ for $1 \le r \le \infty$ and $1 \le s \le \infty$.

The matrix A clearly is the diagonal matrix with the sequence λ on its diagonal. It is also clear that $Ax = \left(\lambda_{mn} x_{mn}\right)_{m,n=0}^{\infty}$ for all $x \in {}^{2}\ell^{r,s}$.

As mentioned before, the compactness of operators will be treated by applying the theory of infinite matrices and the Hausdroff measure of non-compactness, so the next definition and results will be very useful for our work.

We recall the definition of the Hausdroff measure of non-compactness of bounded sets in metric spaces and operators between Banach spaces. Let X be a complete metric space and M_x denote the class of bounded subsets of X. Then the function $\chi : M_x \to [0, \infty)$ defined by

$\chi(Q) = \inf\{\varepsilon > 0 \mid Q$ can be covered by finitely many open balls of radii $< \varepsilon\}$ is called the Hausdroff measure of non-compactness; $\chi(Q)$ is called the Hausdroff measure of non-compactness of the set $Q \in M_X$. Let χ_1 and χ_2 be Hausdroff measures of non-compactness on the Banach-spaces X and Y. An operator $L: X \rightarrow Y$ is said to be (χ_1, χ_2)–bounded if $L(Q) \in M_Y$ for all $Q \in M_X$ and there exists a non-negative real number c such that

$$\chi_2 (L(Q))c.\chi_1(Q) \text{ for all } Q \in M_X. \tag{9}$$

If an operator L is (χ_1, χ_2)–bounded, then the number

$$\|L\|_\chi = \inf\{c \geq 0: (1.9) \text{ holds}\}$$

is called the Hausdroff measure of non-compactness of L.

Theorem: ([8, Theorem 2.25]) Let X and Y be Banach spaces, $A \in (X, Y)$, and $S_X = \{x \in X \mid \|x\| = 1\}$ and $\overline{B}_X = \{x \in X \mid \|x\| \leq 1\}$ denote the unit sphere and closed unit ball in X. Then the Hausdroff measure of non-compactness of the operator L_A, denoted by $\|L_A\|_\chi$, is given by

$$\|L_A\|_\chi = \chi(L_A(\overline{B}_X)) = \chi(L_A(S_X)).$$

Theorem 1.4: ([5, Lemma 2.1]) Let $Q \in M_{\ell^{p,q}}$ $(p \in [1, \infty], q[1, \infty))$, and let R_n: ${}^2\ell^{p,q} \rightarrow {}^2\ell^{p,q}$ for n = 0, 1, 2, … be the operator defined by $R_{mn}(x) = x - x^{[mn]}$ for all $x = (x_{i,j})_{i,j=0}^{\infty} \in {}^2\ell^{p,q}$. Then we have

$$\chi(Q) = \lim_{m,n\to\infty} \left(\sup_{x\in Q} \|R_{mn}(x)\| \right).$$

Theorem 1.5 (Goldenštein, Gohberg, Markus): ([8, Theorem 2.23]) Let X be a Banach space with Schauder basis $\{e_1, e_2, \ldots, e_n\}$. Q be a bounded subset of X and $P_{m,n}$: $X \rightarrow X$ be the projector onto the linear space of $\{e_1, e_2, \ldots, e_n\}$. Then we have

$$\frac{1}{a} \lim_{m+n\to\infty} \sup \left(\sup_{x\in Q} \|(I - P_{m,n})x\| \right) \leq \chi(Q) \leq \lim_{m+n\to\infty} \sup \left(\sup_{x\in Q} \|(I - P_{m,n})x\| \right),$$

where $a = \lim \sup_{m+n\to\infty} \|I - P_{m,n}\|$.

Main Results

Let m and n be non-negative integers. We write $I_{(i,j,m,n)} = I_{(i,j)} \backslash (0, 1, 2, \ldots, m, n)$ and $\lambda^{(mn)} = R_{mn}(\lambda)$. So the operator $T_\lambda^{(mn)}$ is associated with the diagonal matrix $A^{(mn)}(\lambda)$ which is obtained from the diagonal matrix $A(\lambda)$ by replacing $\begin{bmatrix} \lambda_{00} & \lambda_{01} & \cdots & \lambda_{0n} \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{m1} & \lambda_{m2} & \cdots & \lambda_{mn} \end{bmatrix}$, by 0.

Theorem 2.1: Let r, s, u, v, p, q be as in Theorem 1.1. Then we have for $\lambda \in {}^2\ell^{p,q}$.

$$\|T_\lambda\|_\chi = \lim_{m+n\to\infty}\left(\sum\sum_{2\le i+j\le N}\left(\sum\sum_{m'n'\in I(m,n,i,j)}|\lambda_{m'n'}|^p\right)^{q/p}\right)^{1/q} \quad \text{if } v<\infty \text{ and } v<s \text{ and } r>u;$$

$$\|T_\lambda\|_\chi = \lim_{m+n\to\infty}\left(\sum\sum_{2\le i+j\le N}\left(\sup_{m'n'\in I(m,n,i,j)}|\lambda_{m'n'}|^p\right)^{q}\right)^{1/q} \quad \text{if } v<\infty \text{ and } v<s \text{ and } r\le u;$$

$$\|T_\lambda\|_\chi = \lim_{m+n\to\infty}\left(\sup_{i,j}\left(\sup_{m'n'\in I(m,n,i,j)}|\lambda_{m'n'}|^p\right)^{p}\right)^{1/p} \quad \text{if } v<\infty \text{ and } v\ge s \text{ and } r>u;$$

$$\|T_\lambda\|_\chi = \lim_{m+n\to\infty} \sup|\lambda_{mn}| \text{ if } v<\infty \text{ and } v\ge s \text{ and } r\le u;$$

$$0\le \|T_\lambda\|_\chi \le \lim_{m+n\to\infty}\left\|\lambda^{(mn)}\right\|_{p,q} \quad \text{if } v=\infty.$$

Proof: We write $K = \overline{B}_{\ell^{r,s}}$, for short, and denote by A the diagonal matrix that represents the operator $T(\lambda)$. First, we consider the case $v<\infty$. The subcases are $v<s$ and $v\ge s$.
We assume $v<s$. If $r>u$, then we have by Theorem 1.4

$$\|T_\lambda\|_\chi = \chi(L_A(K)) = \lim_{m+n\to\infty}\sup\left\|R_{m,n}(Ax)\right\| = \lim_{m+n\to\infty}\sup_{x\in K}\left\|\lambda.x-(\lambda.x)^{[m,n]}\right\|$$

$$= \lim_{m+n\to\infty}\sup_{x\in K}\left\|T_\lambda^{(mn)}(x)\right\| = \lim_{m+n\to\infty}\left\|\lambda^{(m,n)}\right\|_{p,q}$$

$$= \lim_{m+n\to\infty}\left(\sum\sum_{2\le i+j\le N}\left(\sum\sum_{2\le m'n'\in I(m',n',m,n)}|\lambda_{m'n'}|^p\right)^{q/p}\right)^{1/q}$$

If $r\le u$, then $p=\infty$ and once again we can apply Theorem 1.4, and obtain in the same way as in the previous case

$$\|T_\lambda\|_\chi = \lim_{m+n\to\infty}\left\|\lambda^{(m,n)}\right\|_{\infty,q} = \lim_{m+n\to\infty}\left(\sum\sum_{2\le i+j\le N}\left(\sup_{m'n'\in I(i,j,m,n)}|\lambda_{m'n'}|\right)^{q}\right)^{1/q}.$$

Now we assume $v\ge s$.
The fact that $v\ge s$ means that $q=\infty$. Since v is still less than infinity, we can apply Theorem 1.4 and obtain as in the case $v>s$ the following subcases and corresponding results.
For $r>u$, we have

$$\|T_\lambda\|_\chi = \lim_{m+n\to\infty} \left\|\lambda^{(m,n)}\right\|_{p,\infty} = \lim_{m+n\to\infty} \left(\sup_{i,j} \left(\sum\sum_{m'n'\in I(i,j,m,n)} |\lambda_{m'n'}| \right)^p \right)^{1/p}.$$

Similarly, for r $\leq$ u, we actually obtain p = q = ∞, and so

$$\|T_\lambda\|_\chi = \lim_{m+n\to\infty} \left\|\lambda^{(m,n)}\right\|_{\infty,\infty} = \lim_{m+n\to\infty} \sup |\lambda_{mn}|.$$

We have covered all cases where we were able to apply Theorem 1.4, and established identities for the Hausdroff measure of non-compactness of the operator T_λ. In the case of v = ∞, we will not be able to apply Theorem 1.4. In this case, we will only give estimates for the Hausdroff measure of non-compactness of the operator. As a consequence of this, we will not be able to give necessary and sufficient conditions for the compactness of the operator T_λ as in the case v < ∞.

We assume v = ∞, and that K is unit sphere in ${}^2\ell^{r,s}$, r, s $\in$ [1, ∞]. We define the operators P_{mn}, R_{mn}: ${}^2\ell^{u,\infty} \to {}^2\ell^{u,\infty}$ (n = 0, 1, …) by $P_{mn}(x) = x^{[mn]}$ and $R_{mn}(x) = x - x^{[mn]}$ for $x = (x_{m'n'})_{m',n'=0}^{\infty} \in {}^2\ell^{u,\infty}$. Since $L_A(K) \subset P_{m,n}(L_A(K)) + R_{m,n}(L_A(K))$, it follows from the elementary properties of the function χ [8, Theorem 2.12] that

$$\chi(L_A(K)) \leq \chi(P_{m,n}(L_A(K))) + \chi(R_{m,n}(L_A(K))) = \chi(R_{m,n}(L_A(K))) \leq \sup_{x\in K} \|R_{m,n}(L_A(x))\|.$$

Taking into account the special form of the infinite matrix A associated with the operator T_λ, we obtain

$$\sup_{x\in K} \|R_{m,n}(L_A(x))\| = \sup_{x\in K} \left\|\lambda.x - (\lambda.x)^{[m,n]}\right\| = \sup_{x\in K} \left\|T_\lambda^{[m,n]}(x)\right\| = \left\|T_\lambda^{[m,n]}\right\| = \left\|\lambda^{(m,n)}\right\|_{p,q}.$$

It is clear that in this case we have q = ∞ (either s = v = ∞, or s < v = ∞). The sub-cases which can be considered are r > u and r $\leq$ u; they can be treated in the same way as before. Hence, we obtain $0 \leq \|T_\lambda\|_\chi \leq \lim_{m+n\to\infty} \left\|\lambda^{(m,n)}\right\|_{p,q}$ if v = ∞, that is,

$$0 \leq \|T_\lambda\|_\chi \leq \lim_{m+n\to\infty} \sup_{i,j} \left(\sum\sum_{m',n'\in I(i,j,m,n)} |\lambda_{m'n'}|^p \right)^{1/p}, \text{ if } v = \infty \text{ and } r > u;$$

$$0 \leq \|T_\lambda\|_\chi \leq \lim_{m+n\to\infty} \sup_{m',n'\in I(i,j,m,n)} |\lambda_{m'n'}|, \text{ if } v = \infty \text{ and } r \leq u.$$

All considered cases imply the following corollary by [8, Corollary 2.26 (2.58)].

Corollary: Let r, s, u, v, p, q be as in Theorem 1.1. Then we have for $\lambda \in {}^2\ell^{p,q}$.

$$T_\lambda \text{ is compact if and only if } \lim_{m+n\to\infty} \left(\sum\sum_{2\leq i+j\leq N} \left(\sum\sum_{m',n'\in I(i,j,m,n)} |\lambda_{m'n'}| \right)^{q/p} \right)^{1/q} = 0 \text{ if } v < \infty \text{ and } v < s \text{ and } r > u;$$

T_λ is compact if and only if $\lim\limits_{m+n\to\infty}\left(\sum\sum\limits_{2\le i+j\le N}\left(\sup\limits_{m',n'\in I(i,j,m,n)}|\lambda_{m'n'}|\right)^q\right)^{1/q} = 0$ if $v<\infty$ and $v<s$ and $r\le u$;

T_λ is compact if and only if $\lim\limits_{m+n\to\infty}\left(\sum\sum\limits_{2\le i+j\le N}\left(\sup\limits_{m',n'\in I(i,j,m,n)}|\lambda_{m'n'}|\right)^p\right)^{1/p} = 0$ if $v<\infty$ and $v<s$ and $r\le u$;

T_λ is compact if and only if $\lim\limits_{m+n\to\infty}\sup|\lambda_{mn}| = 0$ if $v<\infty$ and $v\ge s$ and $r\le u$;

If $\lim\limits_{m+n\to\infty}\left\|\lambda^{(m,n)}\right\|_{p,q} = 0$ and $v=\infty$, then T_λ is compact.

Let us remark that in the case when $v=\infty$, we obtain only a sufficient condition for the compactness of the operator T_λ. The application of the Goldenštein-Gohberg-Markus theorem only gives sufficient conditions for the compactness of the operator T_λ associated with the matrix A when ${}^2\ell^{u,v}$ has no Schauder basis. In our paper, that is the case $v=\infty$, that is $T_\lambda: {}^2\ell^{r,s}\to {}^2\ell^{u,\infty}$. But we will give an improvement for a few sub-cases.

Theorem: Let r,s,u,v,p,q be as in Theorem 1.1 r', s', u', p', q' be the conjugate numbers of r,s,u,p,q, $v=\infty$ and $s\ne\infty$. Then we have for $T_\lambda: {}^2\ell^{r,s}\to {}^2\ell^{u,\infty}$:

T_λ is compact if and only if $\lim\limits_{m+n\to\infty}\left(\sup\limits_{i,j}\left(\sum\sum\limits_{m',n'\in I(i,j,m,n)}|\lambda_{mn}|^{p'}\right)^{1/p'}\right) = 0$ if $1<s'<\infty$ and $r'<u'$;

T_λ is compact if and only if $\lim\limits_{m+n\to\infty}\sup|\lambda_{mn}| = 0$ if $1<s'<\infty$ and $u'\le r'$;

If $\lim\limits_{m+n\to\infty}\left\|\lambda^{(m,n)}\right\|_{p',q'} = 0$ and $s=1$, then T_λ is compact.

Proof

We consider the case $v=\infty$. Then ${}^2\ell^{u',v'}={}^2\ell^{u',1}$ is a BK-space with AK. Hence, for $s<\infty$ we can apply [10, Theorem 8.3.9] and obtain the following: $A\in({}^2\ell^{r,s},{}^2\ell^{u,\infty})$ if and only if $A^T\in({}^2\ell^{u',1},{}^2\ell^{r',s'})$ where A^T is transpose matrix of A. Since $A = A(\lambda)$ is the diagonal matrix with the sequence λ on its diagonal, we have $(A(\lambda))^T = A(\lambda)$. Further, applying [9, Theorem 3], we have that the operator T_λ associated with the matrix $A\in({}^2\ell^{r,s},{}^2\ell^{u,\infty})$ is compact if and only if the operator T_λ^T associated with the transpose matrix $A^T\in({}^2\ell^{u',1},{}^2\ell^{r',s'})$ is compact. Since $s\ne\infty$, we have that $s'\ne 1$, that is, $1<s'\le\infty$. Now, the results follow directly from Theorem 1 and Corollary 2 having in mind the definition of the matrix $A = A(\lambda)$ and the following table:

Before	$\mapsto$	Now
r	$\mapsto$	u'
s	$\mapsto$	1
u	$\mapsto$	r'
$v=\infty$	$\mapsto$	s'

The only case which can not be improved, that is, only a sufficient condition can be defined is when $v = s = \infty$.

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